

Approximation Algorithms

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Outline of the tutorial

Today:

- **Approximation algorithms:** an example and definitions
- **Clustering problems:** k -center and k -median
- **Bottleneck problems:** 2-approximation for k -center
- **Local search:** $(3 + \epsilon)$ -approximation for k -median

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Tomorrow:

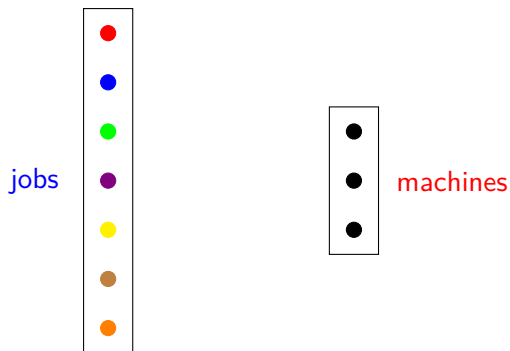
- **Probabilistic strategies:** 0.5-approximation for MaxSAT
- **Linear programming:** 0.63-approximation for the MaxSAT
- **Mixed strategies:** 0.75-approximation for the MaxSAT
- Closing remarks

Scheduling in identical machines

Given: m machines

n jobs

processing time t_i of job i ($i = 1, \dots, n$)

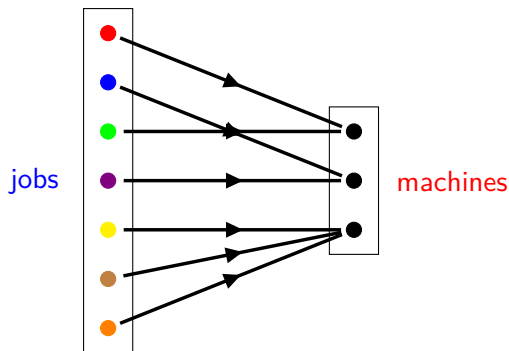


Scheduling in identical machines

Given: m machines

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processing time t_i of job i ($i = 1, \dots, n$)



a **scheduling** is a **partition** $\{M_1, \dots, M_m\}$ of $\{1, \dots, n\}$.

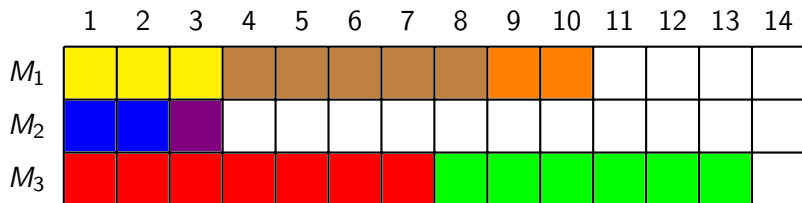
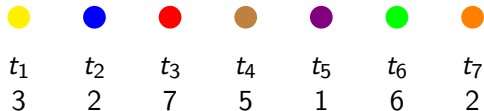
Example 1

$m = 3$ and $n = 7$

						
t_1	t_2	t_3	t_4	t_5	t_6	t_7
3	2	7	5	1	6	2








Example 1

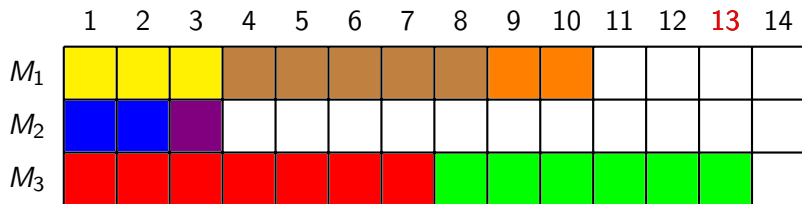
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Example 1

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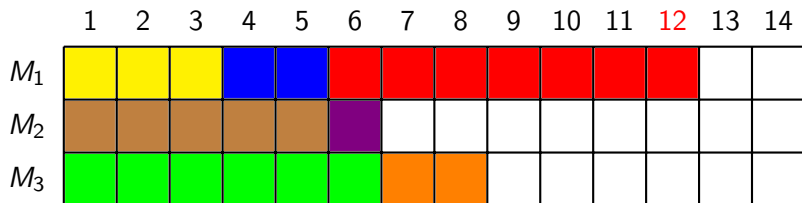
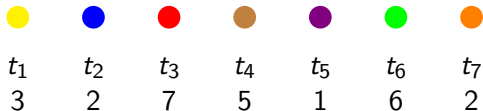
						
t_1	t_2	t_3	t_4	t_5	t_6	t_7
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partition $\{\{1, 4, 7\}, \{2, 5\}, \{3, 6\}\} \Rightarrow \text{makespan} = 13$

Example 2

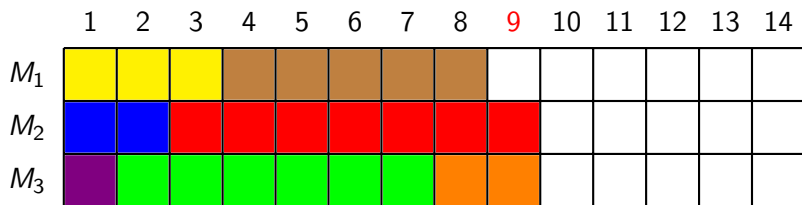
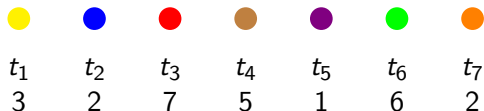
$m = 3$ and $n = 7$



partition $\{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\} \Rightarrow \text{makespan} = 12$

Problem

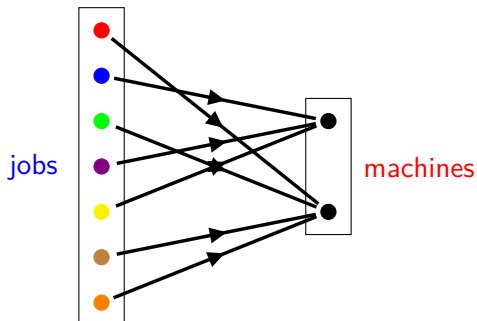
Find a scheduling with **minimum** makespan.



partition $\{\{1, 4\}, \{2, 3\}, \{5, 6, 7\}\} \Rightarrow \text{makespan} = 9$

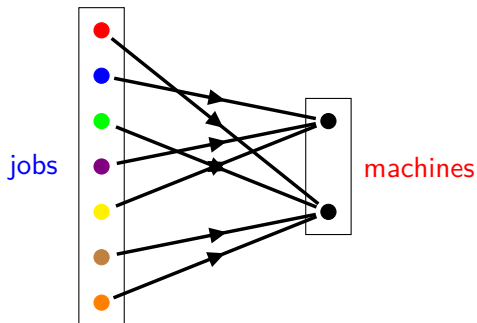
Hardness

Scheduling on two machines: given n and t ,
find a scheduling for **two machines** with minimum makespan.



Hardness

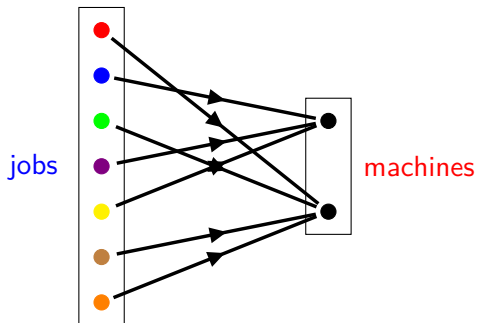
Scheduling on two machines: given n and t ,
find a scheduling for **two machines** with minimum makespan.



Partition: Given a set S numbers,
decide if there is a subset $X \subseteq S$ such that $\sum_{s \in X} s = \sum_{s \in S \setminus X} s$.

Hardness

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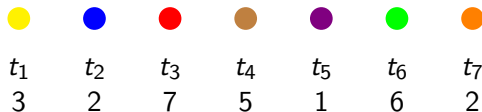
Even this particular case is **NP-hard**, that is,
if there is a polynomial-time algorithm for this case, then $P = NP$.

Graham's algorithm

Assign each job, one by one, to the first available **machine**.

Graham's algorithm

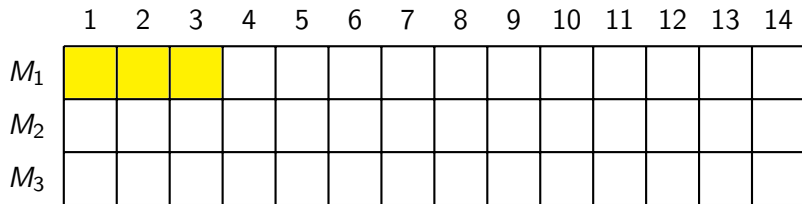
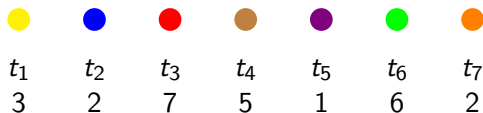
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	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M_1														
M_2														
M_3														

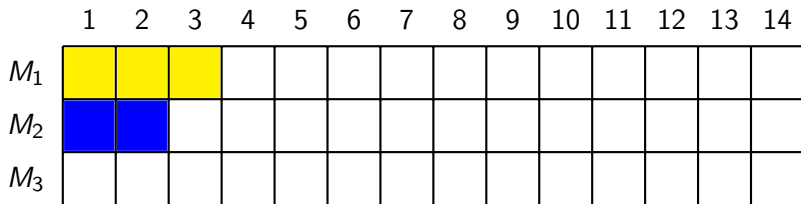
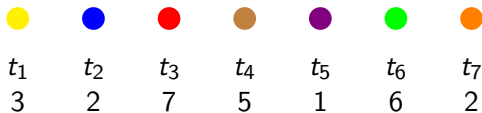
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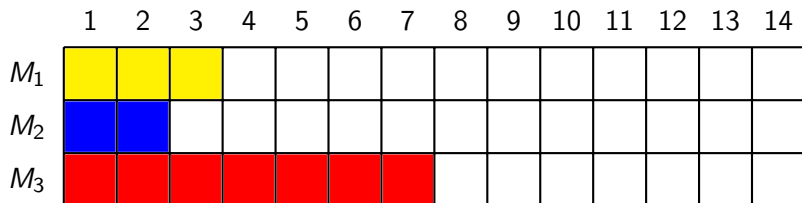
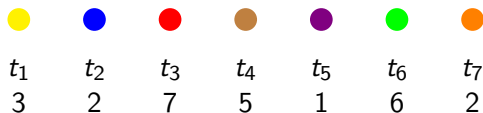
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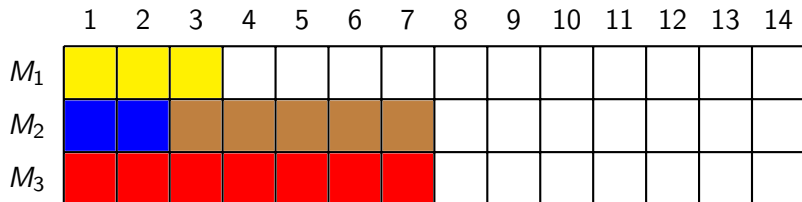
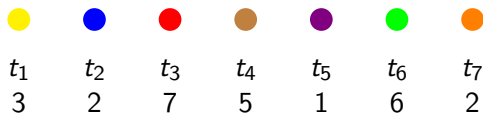
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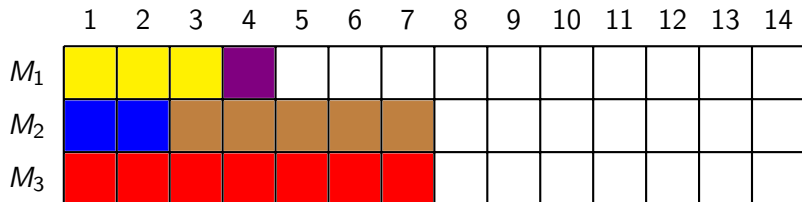
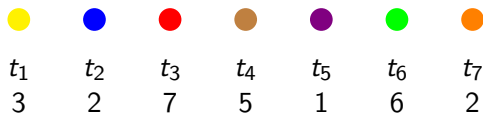
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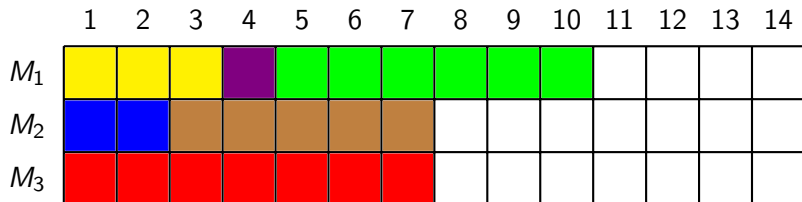
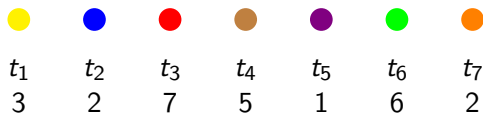
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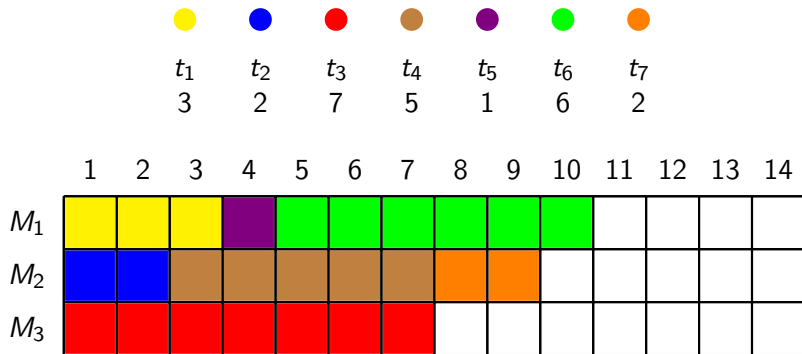
Graham's algorithm

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Graham's algorithm

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Graham's algorithm is polynomial.

How bad can the makespan be?

Bounds on OPT

OPT = minimum makespan

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- Largest processing time of a job:

$$\text{OPT} \geq \max\{t_1, t_2, \dots, t_n\}$$

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- Balanced distribution:

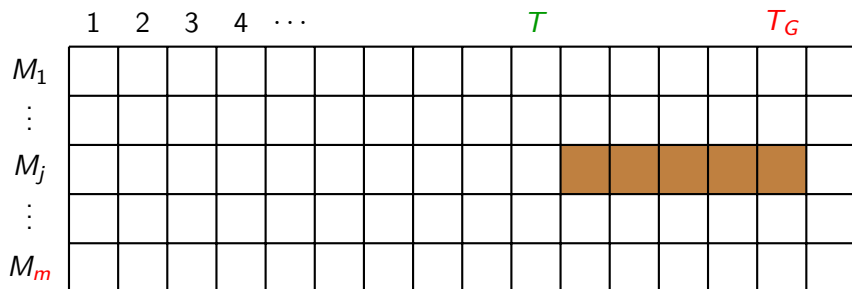
$$\text{OPT} \geq \frac{t_1 + t_2 + \dots + t_n}{m}$$

Makespan of Graham's scheduling

T_G : makespan of the algorithm

job i : job that finishes at time T_G

time T : time previous to the starting time of job i

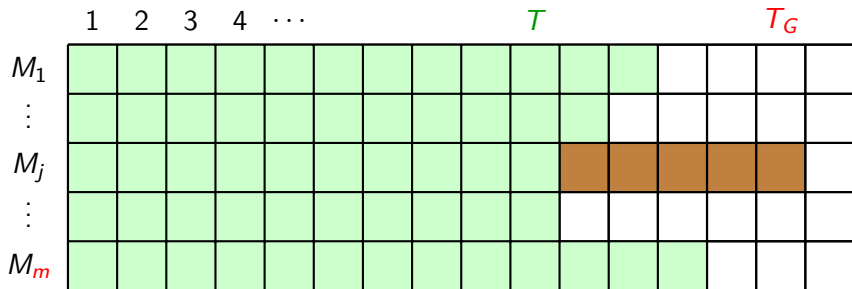


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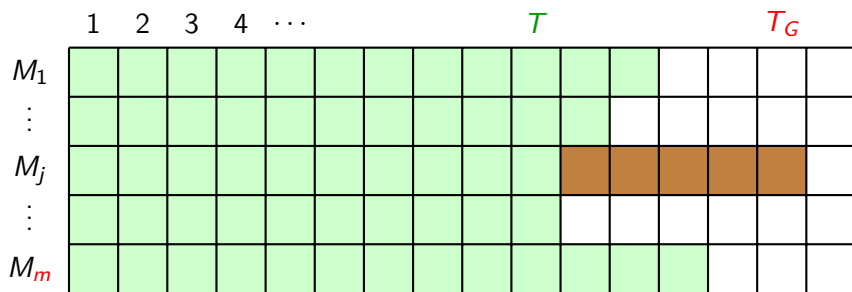
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$$T \cdot m < t_1 + \dots + t_n \quad \Rightarrow \quad T < \frac{t_1 + \dots + t_n}{m} \leq \text{OPT}$$

Quality of Graham's scheduling



$$\begin{aligned} T_G &= T + t_j \\ &< \text{OPT} + \max\{t_1, \dots, t_n\} \\ &\leq \text{OPT} + \text{OPT} \\ &= 2 \text{OPT} \end{aligned}$$

Approximation algorithm

Context:

- Π : optimization problem (minimization)
- $\text{cost}(S, I)$: cost of the feasible solution S for instance I of Π
- $\text{OPT}(I)$: minimum cost of a feasible solution for instance I of Π

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Algorithm for Π :

given any instance I for Π , produces a feasible solution for I .

$A(I)$: solution produced by algorithm A on instance I

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Approximation algorithm

if A is polynomial and there exists a number $\alpha \geq 1$ such that

$$\text{cost}(A(I), I) \leq \alpha \text{OPT}(I) \text{ for every instance } I \text{ of } \Pi,$$

then A is an **α -approximation**.

Graham's algorithm

Input: positive integers m and n , and an array $t[1..n]$

Output: a scheduling of $\{1, \dots, n\}$ in m machines.

Algorithm Graham (m, n, t)

- 1 for $j := 1$ to m do
- 2 $M_j := \emptyset$
- 3 $T_j := 0$ ▷ available instant for machine j
- 4 for $i := 1$ to n do
- 5 let k be such that T_k is minimum
- 6 $M_k := M_k \cup \{i\}$ $T_k := T_k + t_i$
- 7 return $\{M_1, \dots, M_m\}$

Graham's algorithm is a **2-approximation**.

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Exercise 1:

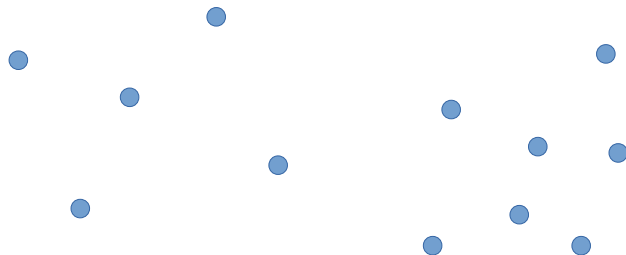
What if we schedule the jobs in decreasing order of the processing time?

Clustering problems

Classical k -center

Given:

- a positive integer k ,
- a set V of elements, and
- a function $d : V \times V \rightarrow \mathbb{Q}^+$,



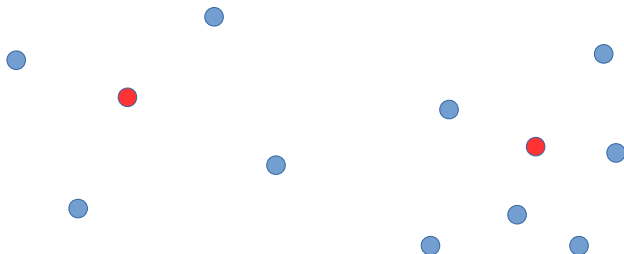
Clustering problems

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find a set $S \subseteq V$ with $|S| = k$ that ...



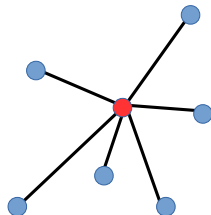
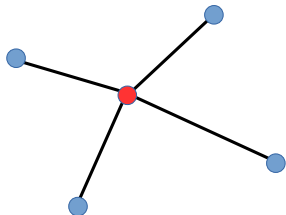
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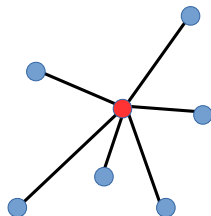
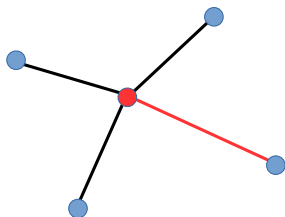
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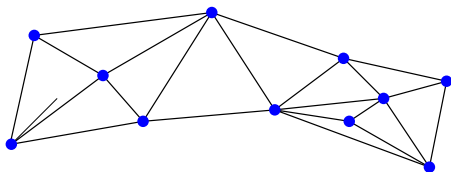
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Hardness

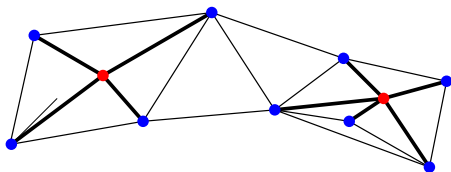
Graph G



Dominating set: set S of vertices of G such that each vertex of G is in S or has a neighbor in S .

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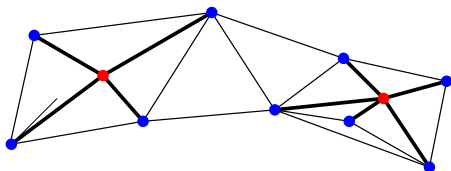
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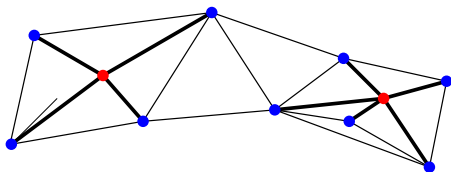


Dominating set: set S of vertices of G such that each vertex of G is in S or has a neighbor in S .

Reduction to a k -center instance: take the same k , $V = V(G)$ and $d(x, y) = 1$ if x and y are adjacent in G , and $d(x, y) = M$ otherwise.

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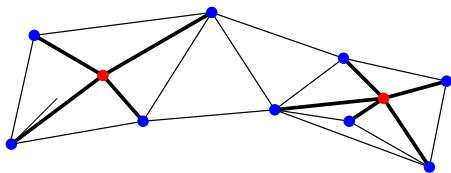
Reduction to a k -center instance: take the same k , $V = V(G)$ and $d(x, y) = 1$ if x and y are adjacent in G , and $d(x, y) = M$ otherwise.

Theorem

There is a dominating set of size k in G if and only if there is a k -center solution of radius 1 for the instance (k, V, d) .

Inapproximability

Graph G



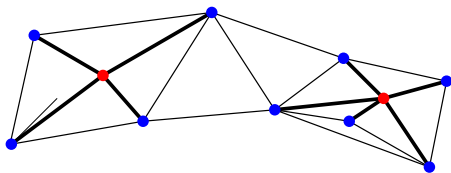
The k -center instance is $V = V(G)$ and $d(x, y) = 1$ if x and y are adjacent in G , and $d(x, y) = M$ otherwise.

Hard even to approximate:

An α -approximation for k -center with $\alpha < M$ solves dominating set.

Inapproximability

Graph G



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Hard even to approximate:

An α -approximation for k -center with $\alpha < M$ solves dominating set.

Theorem

There is no α -approximation for the k -center problem, unless $P = NP$.

Too hard to approximate?

What to do?

Restrict attention to specific classes of instances.

Too hard to approximate?

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Restrict attention to specific classes of instances.

A function $d : V \times V \rightarrow \mathbb{Q}^+$ is a **metric** if, for every $x, y, w \in V$,

- $d(x, y) = d(y, x)$ (symmetry)
- $d(x, y) \leq d(x, w) + d(w, y)$ (triangle inequality)

Such a function d is called a **distance function**.

Metric instances

If d is a distance function, then the instance is **metric**.

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The **k -center instance built from the dominating set instance:**

$d(x, y) = 1$ if x and y are adjacent in G and $d(x, y) = M$ otherwise.

This k -center instance is metric only if $M \leq 2$.

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Exercise 2:

How does the previous inapproximability result apply to the metric k -center?

Bottleneck problems

- Optimal value is the **length** of an edge

Bottleneck problems

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- We can “**guess**” the optimal value

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Example of bottleneck problem: **k -center**

Instance: positive integer k , set V , and distance function d on V .

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Example of bottleneck problem: **k -center**

Instance: positive integer k , set V , and distance function d on V .

G : complete graph on V with length $\ell(uv) = d(u, v)$ for each edge uv .

Bottleneck problems

Metric k -center: positive integer k and complete graph G on V with length $\ell(uv) = d(u, v)$ for each edge uv .

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Metric k -center: positive integer k and complete graph G on V with length $\ell(uv) = d(u, v)$ for each edge uv .

Idea for an algorithm:

- 1 Sort edges of G by length: $\ell(e_1) \leq \dots \leq \ell(e_m)$.

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- 3 i^* : smallest i such that G_i has a **dominating set of size k** .

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H^2 : the square of H (add edges between vertices at distance 2 in H)

Bottleneck problems

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Idea for an algorithm:

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- 2 Threshold graph G_i : $G[E_i]$ where $E_i := \{e_1, \dots, e_i\}$.
- 3 i^* : smallest i such that G_i has a dominating set of size k ← HARD
- 4 $\ell(e_{i^*})$: radius of optimal k -center solution.

H^2 : the square of H (add edges between vertices at distance 2 in H)

A maximal independent set in a graph is a dominating set.

Bottleneck problems

Metric k -center: positive integer k and complete graph G on V with length $\ell(uv) = d(u, v)$ for each edge uv .

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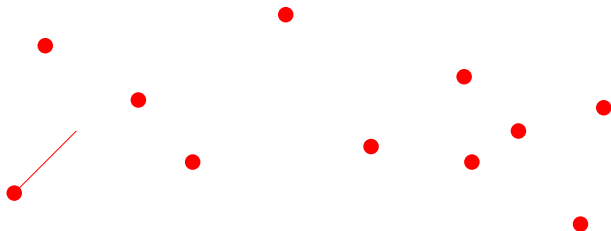
A maximal independent set in G_i^2 is a set of centers in G of radius $2\ell(e_i)$.

Bottleneck problems: metric k -center

Algorithm GHS (k, G, ℓ) \triangleright Gonzalez '85, Hochbaum and Shmoys '85

- 1 $M_0 := V(G)$ $i := 0$
- 2 while $|M_i| > k$
- 3 $i := i + 1$
- 4 Let M_i be a maximal independent set on G_i^2
- 5 return M_i

$k = 2$

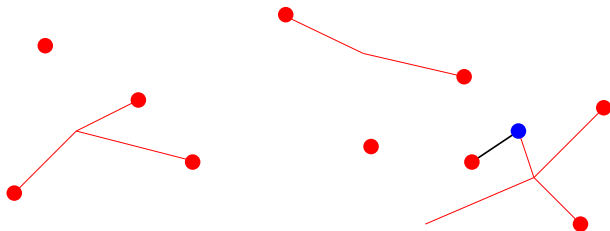


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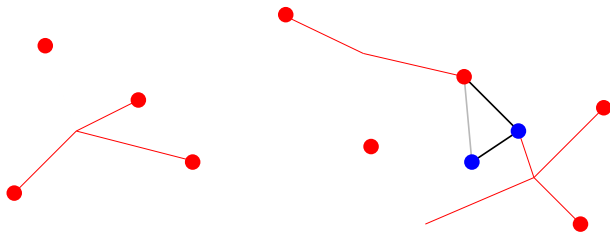


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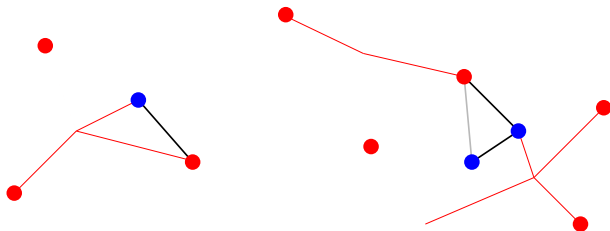


Bottleneck problems: metric k -center

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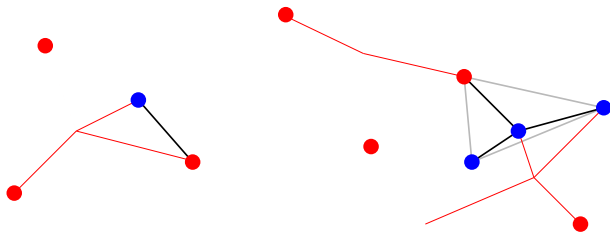


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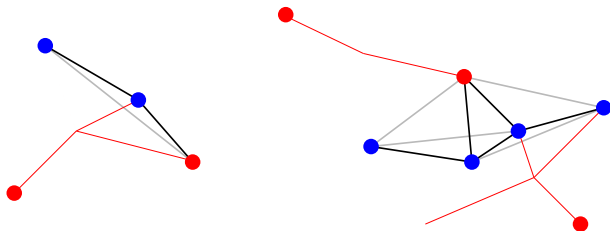


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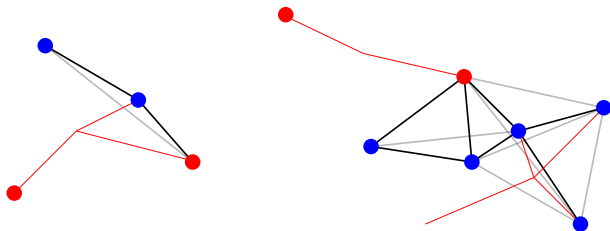


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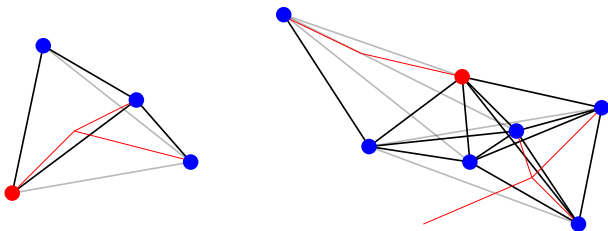


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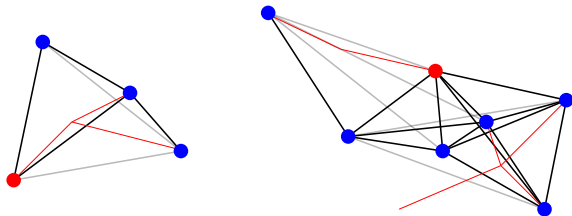


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The radius of M_i is at most $2\ell(e_i)$.

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The radius of M_i is at most $2\ell(e_i)$.

Because G_{i^*} has a dominating set of size k , any maximal independent set in $G_{i^*}^2$ has size at most k .

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Hence the radius of M_i is at most $2\ell(e_i) \leq 2\ell(e_{i^*}) = 2 \text{OPT}$.

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Exercise 3:

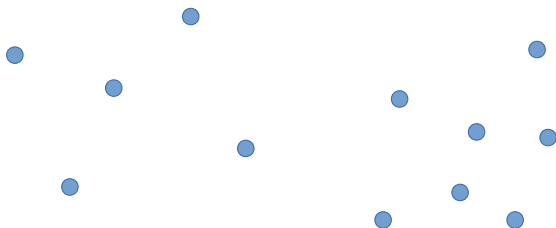
Is there an α -approximation with $\alpha < 2$ for the metric k -center?

Clustering problems

Classical k -median

Given:

- a positive integer k ,
- a set V of elements, and
- a function $d : V \times V \rightarrow \mathbb{Q}^+$,



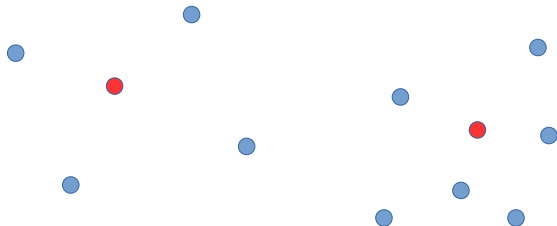
Clustering problems

Classical k -median

Given:

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find a set $S \subseteq V$ with $|S| = k$ that ...



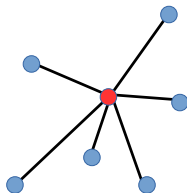
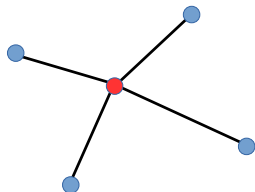
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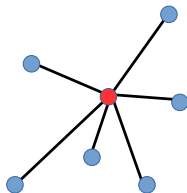
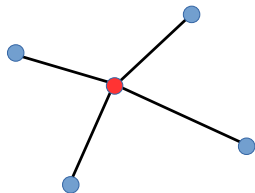
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There is no α -approximation
for constant $\alpha > 1$ unless $P = NP$.

Local search: **metric** k -median

k -median instance: positive integer k , set V , and distance function d .

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Let S be a subset of V of size k . Let $u \in S$ and $v \notin S$.

Local search: metric k -median

k -median instance: positive integer k , set V , and distance function d .

Let S be a subset of V of size k . Let $u \in S$ and $v \notin S$.

Pair (u, v) is an **improving swap** for S if $S' = S - u + v$ has better cost:

$$\sum_{u \in V} \min_{v \in S} d(u, v) > \sum_{u \in V} \min_{v \in S'} d(u, v).$$

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Algorithm AGKMMP (k, V, d)

▷ Arya et al. '01

- 1 let S be an arbitrary set of k elements of V
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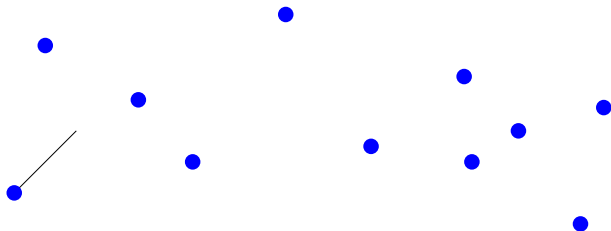
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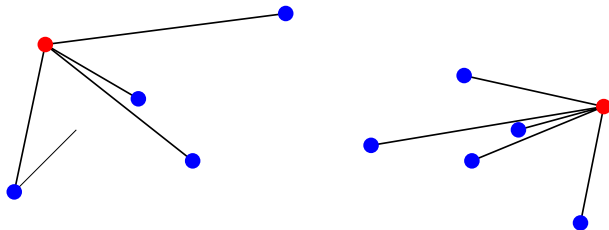
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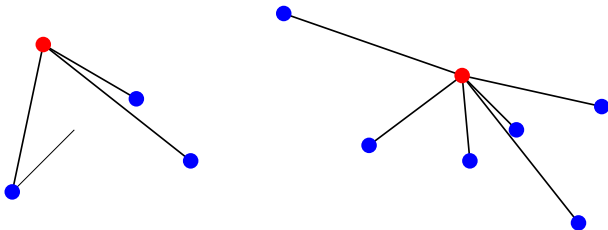
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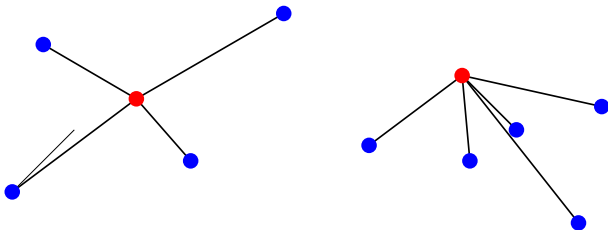
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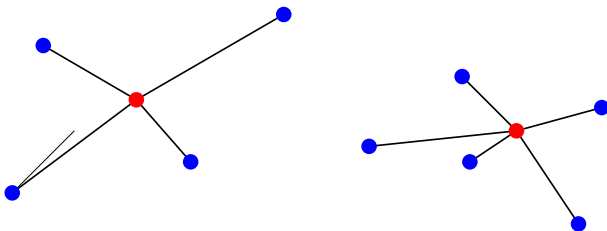
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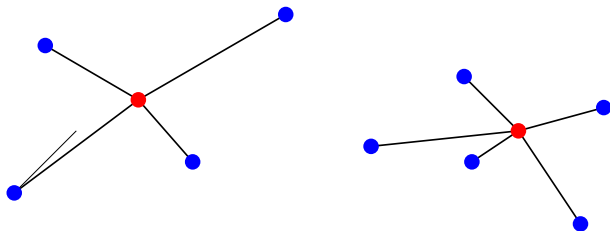
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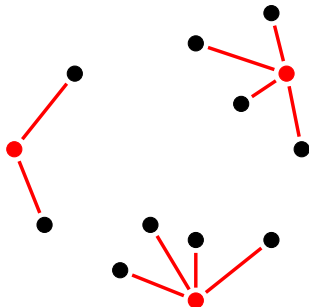
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This is a 5-approximation!

Sketch of the approximation analysis

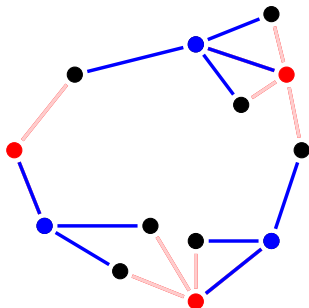
Let S^* be an optimal solution and $N^*(o)$ be the clients of o in S^*



Sketch of the approximation analysis

Let S^* be an optimal solution and $N^*(o)$ be the clients of o in S^*

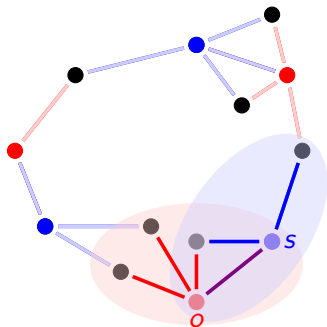
Let S be the output of the algorithm and $N(s)$ be the clients of s in S .



Sketch of the approximation analysis

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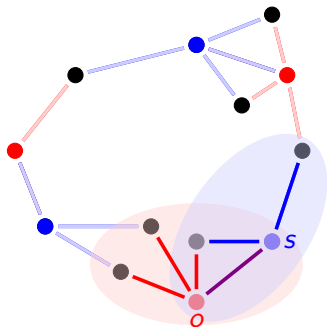


$s \in S$ captures $o \in S^*$ if $|N^*(o) \cap N(s)| > |N^*(o)|/2$.

Sketch of the approximation analysis

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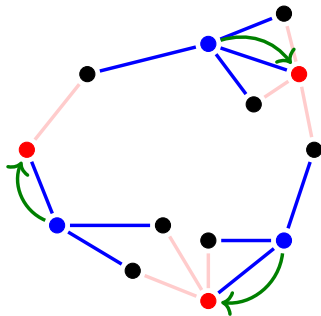
$s \in S$ captures $o \in S^*$ if $|N^*(o) \cap N(s)| > |N^*(o)|/2$.

Each $o \in S^*$ is captured by at most one element from S .

Sketch of the approximation analysis

$s \in S$ captures $o \in S^*$ if $|N^*(o) \cap N(s)| > |N^*(o)|/2$.

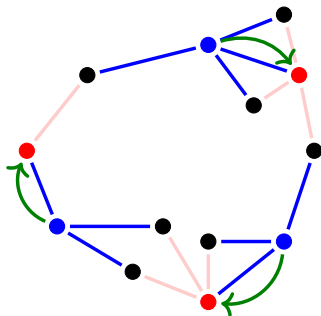
Assume that each $s \in S$ captures exactly one element o from S^* .



Sketch of the approximation analysis

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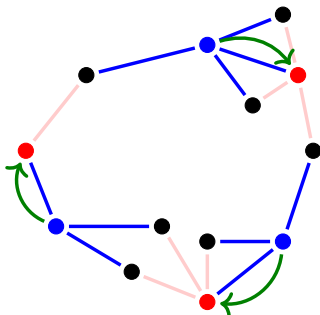


In this case, let us prove that $\text{cost}(S) \leq 3 \text{cost}(S^*)$.

Sketch of the approximation analysis

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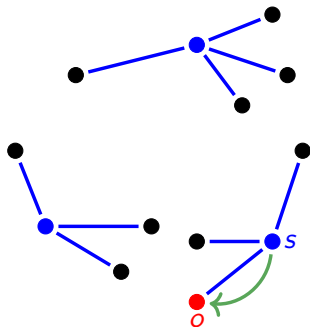
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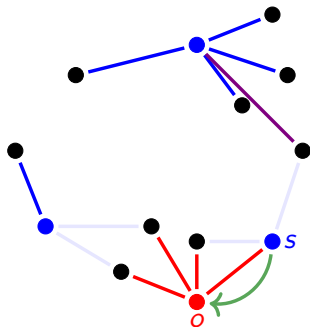
Because (s, o) is not an improving swap, $\text{cost}(S - s + o) \geq \text{cost}(S)$.

Sketch of the approximation analysis



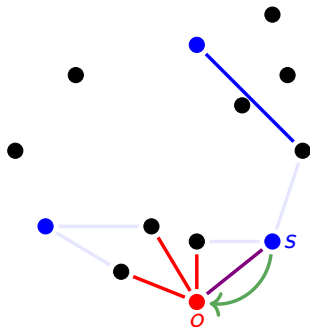
$$\text{cost}(S - s + o) \leq ???$$

Sketch of the approximation analysis



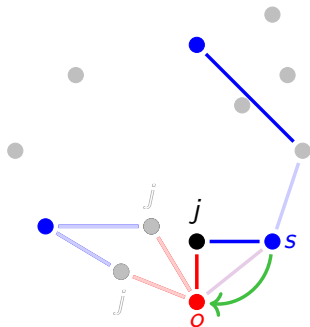
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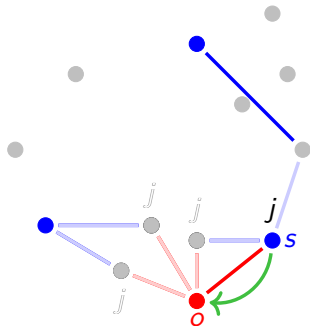
$$\text{cost}(S - s + o) \leq \text{cost}(S) + ???$$

Sketch of the approximation analysis



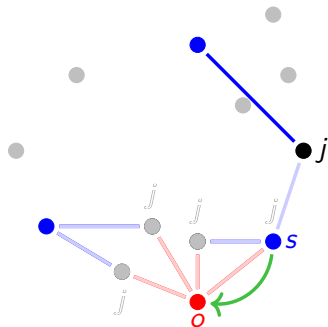
$$\text{cost}(S - s + o) \leq \text{cost}(S) + \sum_{j \in N^*(o)} (o_j - s_j) +$$

Sketch of the approximation analysis



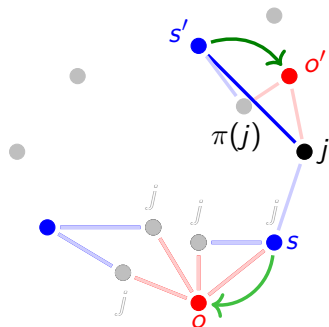
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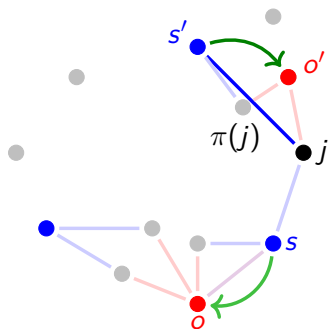
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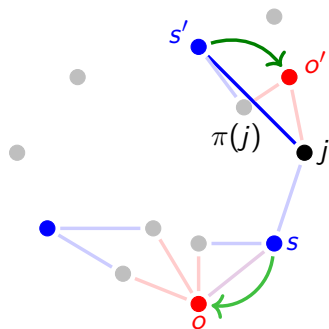
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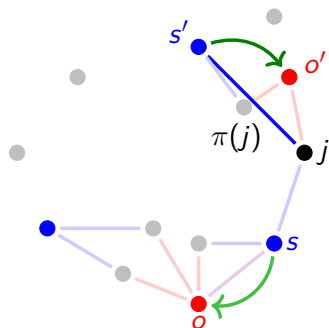
$$\text{cost}(S - s + o) \leq \text{cost}(S) + \sum_{j \in N^*(o)} (o_j - s_j) + \sum_{j \in N(s) \setminus N^*(o)} (o_j + o_{\pi(j)} + s_{\pi(j)} - s_j).$$

Sketch of the approximation analysis



$$\text{cost}(S - s + o) \leq \text{cost}(S) + \sum_{j \in N^*(o)} (o_j - s_j) + \sum_{j \in N(s)} (o_j + o_{\pi(j)} + s_{\pi(j)} - s_j).$$

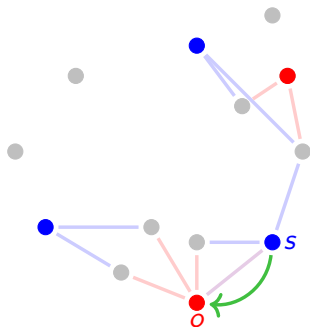
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Permutation π is selected using that $|N^*(o') \cap N(s')| > |N^*(o)|/2$.

Sketch of the approximation analysis



$$\text{cost}(S - s + o) \leq \text{cost}(S) + \sum_{j \in N^*(o)} (o_j - s_j) + \sum_{j \in N(s)} (o_j + o_{\pi(j)} + s_{\pi(j)} - s_j).$$

But $\text{cost}(S - s + o) \geq \text{cost}(S)$, so

$$\sum_{j \in N^*(o)} (o_j - s_j) + \sum_{j \in N(s)} (o_j + o_{\pi(j)} + s_{\pi(j)} - s_j) \geq 0.$$

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Exercise 4:

Argue that each $s \in S$ captures at most two elements from S^* and

derive that $\sum_{j \in N^*(o)} (o_j - s_j) + 2 \sum_{j \in N(s)} (o_j + o_{\pi(j)} + s_{\pi(j)} - s_j) \geq 0$.

That is all for today!

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Homework for tomorrow:

- 1 Sort the jobs in decreasing order of the processing time before running Graham's algorithm. Can you prove an approximation ratio better than 2 for this algorithm?
- 2 How does the inapproximability result for k -center apply to metric instances?
- 3 Is there an α -approximation with $\alpha < 2$ for the metric k -center?
- 4 Argue that each $s \in S$ captures at most two elements from S^* and derive that $\sum_{j \in N^*(o)} (o_j - s_j) + 2 \sum_{j \in N(s)} (o_j + o_{\pi(j)} + s_{\pi(j)} - s_j) \geq 0$.

Satisfiability

Boolean formulas

v_i : boolean variable

\bar{v}_i : negation of the boolean variable v_i

literal: a variable or its negation

clause: disjunction (OR) of literals, as for instance $v_1 \vee \bar{v}_2 \vee v_3$

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Boolean formula in conjunctive normal form (CNF):

$$\phi = (v_1 \vee \bar{v}_2 \vee v_3)(\bar{v}_1 \vee \bar{v}_3)(v_2 \vee v_3 \vee \bar{v}_4 \vee v_5)(\bar{v}_1 \vee v_4 \vee \bar{v}_5)$$

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assignment for ϕ : function that assigns **True** or **False** to each variable in ϕ

To decide whether there exists an assignment that satisfies a CNF formula is NP-complete.

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MAX SAT Problem

Given a CNF formula ϕ ,
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$$\Pr[C \text{ is satisfied}] = 1 - \frac{1}{2^k}$$

Probabilistic $\frac{1}{2}$ -approximation for SAT

Algorithm Johnson (ϕ)

▷ Johnson '74

- 1 let V be the set of variables in ϕ
- 2 for each $v \in V$
- 3 $x_v := \text{RAND}(1/2)$
- 4 return x

▷ gives a (probabilistic) 0.5-approximation

$\text{RAND}(p)$: returns 1 with probability p or 0 with probability $1 - p$.

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Let m be the number of clauses in ϕ .

Then clearly $\text{Exp}[\text{cost}(x)] \geq m/2 \geq \text{OPT}(\phi)/2$.

One more exercise

Proposed algorithm

- 1 let V be the set of variables in ϕ
- 2 $s := \text{RAND}(1/2)$ ▷ unique coin flip
- 3 for each $v \in V$
- 4 $x_v := s$
- 5 return x

$\text{RAND}(p)$: returns 1 with probability p and 0 with probability $1 - p$.

Exercise 5:

Prove that the proposed algorithm is an α -approximation for some α , or argue that the algorithm is not an approximation algorithm.

Integer programming formulations

For a CNF formula ϕ ,
 V is its set of variables.

For a clause C of ϕ ,
 $C_0 := \{v_i : \bar{v}_i \in C\}$ and $C_1 := \{v_i : v_i \in C\}$.

C_0 are the **negative** variables in C and C_1 are the **positive** variables in C .

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Consider the following integer linear program (IP) built from ϕ :

maximize $\sum_{C \in \phi} z_C$
subject to

$$\begin{aligned} \sum_{v \in C_0} (1 - x_v) + \sum_{v \in C_1} x_v &\geq z_C && \text{for every } C \in \phi \\ z_C &\in \{0, 1\} && \text{for every } C \in \phi \\ x_v &\in \{0, 1\} && \text{for every } v \in V \end{aligned}$$

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Solving this IP is equivalent to finding **OPT**(ϕ).

Linear programming and rounding

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Idea

Use the value of $x_v \in [0, 1]$ to decide how to set v to **True** or **False**.

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Algorithm GW (ϕ)

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- 1 solve the LP above obtaining \hat{z} and \hat{x}
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Indeed, \dot{x} satisfies at least $0.63 \sum_{C \in \phi} \hat{z}_C \geq 0.63 \text{OPT}(\phi)$ clauses.

Analysis

For each clause C , let t be the number of literals in C .

Consider the binary random variable

Z_C that is 1 if C is satisfied by \dot{x} and 0 otherwise.

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For non-negative numbers, geometric is smaller than arithmetic mean:

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Hence

$$\text{Exp}[Z_C] \geq 1 - \left(\frac{t - \hat{z}_C}{t} \right)^t.$$

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For each clause C with t literals,
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because $f(z) = 1 - \left(1 - \frac{z}{t}\right)^t$ is concave in the interval $[0, 1]$,
and $f(0) = 0$, so $f(z) \geq z f(1)$, which implies the last inequality.

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because $\left(1 - \frac{1}{t}\right)^t < \frac{1}{e}$ for every $t \geq 1$.

Euler's number $e = 2.71828$, the base of the natural logarithm.

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For each clause C with t literals,
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Note that $\sum_{C \in \phi} Z_C$ is the number of clauses satisfied
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As, for every $C \in \phi$,

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The GW algorithm is a 0.63-approximation for MAXSAT.

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If all clauses have k literals,
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which gets worse as k grows.

So one of the algorithms works better on formulas whose clauses are long,
and the other on formulas whose clauses are short.

Idea

Run both algorithms and output the best solution.

Algorithm Combined (ϕ)

▷ Goemans and Williamson '94

- 1 $x_J := \text{JOHNSON}(\phi)$
- 2 $x_{GW} := \text{GW}(\phi)$
- 3 let s_J be the number of clauses of ϕ satisfied by x_J
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- 5 if $s_J \geq s_{GW}$ then return x_J
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X_{GW} : number of clauses satisfied by GW algorithm.

\mathcal{C}_k : clauses in ϕ with exactly k literals

$$\begin{aligned}\text{Exp}[\max\{X_J, X_{GW}\}] &\geq \text{Exp}\left[\frac{X_J + X_{GW}}{2}\right] \\ &\geq \frac{1}{2} \sum_k \sum_{C \in \mathcal{C}_k} \left(\left(1 - \frac{1}{2^k}\right) + \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \hat{z}_C \right) \\ &\geq \frac{1}{2} \sum_k \sum_{C \in \mathcal{C}_k} \left(1 - \frac{1}{2^k} + 1 - \left(1 - \frac{1}{k}\right)^k \right) \hat{z}_C\end{aligned}$$

Joining ideas

X_J : number of clauses satisfied by Johnson's algorithm.

X_{GW} : number of clauses satisfied by GW algorithm.

\mathcal{C}_k : clauses in ϕ with exactly k literals

$$\begin{aligned}\text{Exp}[\max\{X_J, X_{GW}\}] &\geq \text{Exp}\left[\frac{X_J + X_{GW}}{2}\right] \\ &\geq \frac{1}{2} \sum_k \sum_{C \in \mathcal{C}_k} \left(\left(1 - \frac{1}{2^k}\right) + \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \hat{z}_C \right) \\ &\geq \frac{1}{2} \sum_k \sum_{C \in \mathcal{C}_k} \left(1 - \frac{1}{2^k} + 1 - \left(1 - \frac{1}{k}\right)^k \right) \hat{z}_C \\ &\geq \frac{1}{2} \sum_k \sum_{C \in \mathcal{C}_k} \frac{3}{2} \hat{z}_C\end{aligned}$$

Joining ideas

X_J : number of clauses satisfied by Johnson's algorithm.

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$$\begin{aligned}\text{Exp}[\max\{X_J, X_{GW}\}] &\geq \text{Exp}\left[\frac{X_J + X_{GW}}{2}\right] \\ &\geq \frac{1}{2} \sum_k \sum_{C \in \mathcal{C}_k} \left(\left(1 - \frac{1}{2^k}\right) + \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \hat{z}_C \right) \\ &\geq \frac{1}{2} \sum_k \sum_{C \in \mathcal{C}_k} \left(1 - \frac{1}{2^k} + 1 - \left(1 - \frac{1}{k}\right)^k \right) \hat{z}_C \\ &\geq \frac{1}{2} \sum_k \sum_{C \in \mathcal{C}_k} \frac{3}{2} \hat{z}_C \\ &= \frac{3}{4} \text{OPT}(\phi)\end{aligned}$$

Conclusions

If you like algorithms and the use of smart ideas to design beautiful and efficient algorithms, join the force to study **approximation algorithms**!

Two books on the subject

Approximation Algorithms, by Vazirani

The Design of Approximation Algorithms, by Williamson and Shmoys

THANK YOU!!!