Neutron Star Asteroseismology

Problem Set 1

20/05/2019

1. Plot the Regge-Wheeler potential together with the Zerilli potential for a given value of M and $\ell = 2, 3, 4$, first as a function of r and then as a function of r_*. What can you conclude from these plots?

2. It can be shown that general relativity implies the existence of maximum mass for a compact star. A simple form of this argument gives that $M_0 \leq \frac{1}{2}\left(\frac{3}{8\pi\rho_0}\right)^{1/2}$, where M_0 is the mass of the core restricted to densities $\rho > \rho_0$. Put the factors of c and G back in this equation and scale ρ_0 by the nuclear density to find that $M_0 \lesssim 8.0 M_\odot \left(\frac{2.9\times10^{14}\,\mathrm{g/cm^3}}{\rho_0}\right)^{1/2}$.

3. Most of the binary pulsars have white dwarf stars as companions, and the binary systems are practically circular. Find out a method based on pulsar timing to measure the masses for such systems.

4. Solve the relativistic equations of stellar structure for a static, spherically symmetric star of uniform density. Show that the mass and radius of the star satisfy $R/2M > 9/8$.

5. Computer exercise: Write a simple code to integrate the TOV equation in the case of a polytropic equation of state $p = K\rho^\Gamma$, for given central pressure p_c and polytropic index Γ.

Neutron Star Asteroseismology

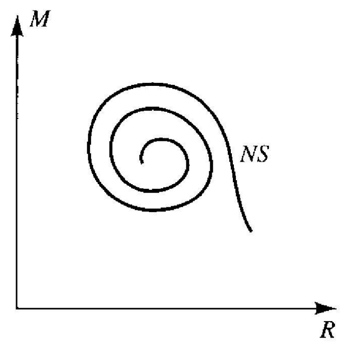
Problem Set 2

21/05/2019

1. Verify that $\omega^2 = 0$ and $\xi = \text{const.} \times r$ is a solution of the radial perturbation equation for $\Gamma_1 = \frac{4}{3}$. What can you conclude from this result?

2. Work through the changes in stability of the modes that occur at the extrema of the mass vs. radius relation for the family of stars in Fig. 24.7. Assume that the curves of squared frequency vs. central density never cross. Show that the changes in stability are as illustrated in Fig. 24.11 and that there are only two ranges of equilibrium stars, as shown.

3. *Stable equilibria beyond neutron stars?* A theorist proposes a new equation of state for matter above nuclear densities and wonders whether it might lead to a new kind of ultra-high-density endstates to stellar evolution beyond neutron and white dwarf stars. You use the equation of state and the relativistic equations of stellar structure to calculate the mass-radius relationship of stars with central density greater than nuclear density that is shown below. The curve represents stable neutron stars at the lowest densities but then spirals around at higher densities. Assuming the lowest-density, largest-radius stars shown are stable, will there be a new family of stable equilibria?



4. Polytropic stars are unstable in Newtonian theory if $\Gamma < \frac{4}{3}$. Consider the influence of small relativistic effects on this stability criterion. Show that the effect is to increase the unstable range of Γ to $\Gamma < \frac{4}{3} + \varepsilon$, where ε may depend on the mass, radius and structure of the star.

5. Computer exercise: Find the lowest few stable pulsation frequencies of a polytrope of your favorite value of index n for $\Gamma_1 = \frac{5}{3}$. Hint:

   (a) Nondimensionalize the eigenvalue equations

   (b) Determine the solution to the hydrostatic equilibrium equation simultaneously

   (c) Investigate analytically the behavior of the solution near $r = 0$ and $r = R$

   (d) A possible numerical method is to integrate out from $r = 0$ and in from $r = R$ to some convenient r inside the star with a guessed value of ω^2, imposing the boundary conditions determined in the previous step. Since you do not know the relative magnitude of these two solutions, start them both with unit magnitude. If you had guessed the correct value for ω^2, the Wronskian of the two solutions at the "matching radius" r would vanish. (Why?) In general, it will be nonzero. Choose another value of ω^2, integrate in from the boundaries to r, and compute the Wronskian again. Now interpolate (or extrapolate) to that value of ω^2 that zeros the Wronskian. Continue to iterate until ω^2 converges to the required number of significant digits.

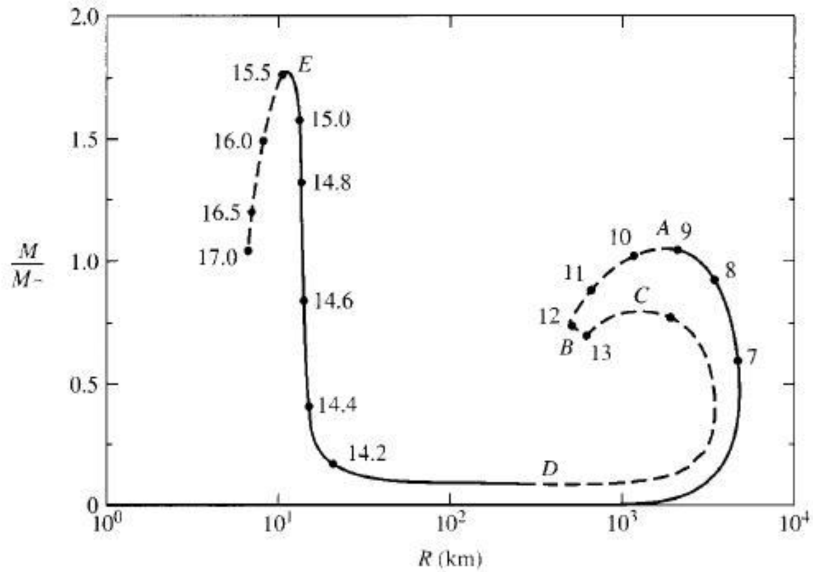


FIGURE 24.7 Mass vs. radius for the nonrotating equilibrium endstates of stellar evolution calculated from the equation of state of matter in its ground state described in Box 24.1 and summarized in Figure 24.6. The curve is the one-parameter family of equilibrium endstates of stellar evolution parametrized by the central density, ρ_c. Values of $\log_{10}[\rho_c(\mathrm{g/cm^3})]$ are indicated along the curve. The extrema are labeled by $A, B, C, \ldots$. The curve begins at the origin with $M \propto R^3$, but this low-density part of the curve is indistinguishable from the horizontal axis on the scales of the figure. There are two regions of stable equilibria indicated by solid lines. Stars with densities below the first maximum of the mass at A are supported by the Fermi pressure of electrons arising from the Pauli exclusion principle. The other family of stable equilibria lies between the second minimum of the mass at D and the third maximum at E. They consist mostly of neutron nuclear matter and are, therefore, called *neutron stars*. The repulsive forces between nucleons are the dominant source of the pressure. The dotted parts of the curve are unstable configurations, which will not exist in nature.

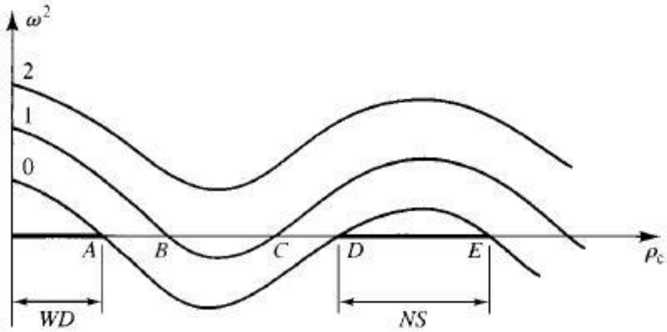


FIGURE 24.11 A schematic representation of the squared frequencies of the lowest three radial modes of a family of nonrotating stars shown in Figure 24.7 as a function of central density ρ_c with 0, 1, and 2 nodes, as labeled. At low densities all modes have positive squared frequency and are stable. At the density A the lowest mode's squared frequency turns negative, and the mode becomes unstable. At B the second mode also becomes unstable. These two modes return to stability in succession at C and D. The lowest mode again becomes unstable at E. Zero-frequency modes that characterize a change in stability occur at these central densities. These are displacements between equilibrium configurations that can occur only at extrema of the M vs. R curve in Figure 24.7, which are labeled by A, B, etc. Stars are stable only when all their modes are stable. This happens in the range of densities from 0 to A corresponding to white dwarfs and D to E corresponding to neutron stars.

Neutron Star Asteroseismology

Problem Set 3

22/05/2019

1. Show that the axial perturbation equation for a star reduces to the Regge-Wheeler equation outside of the star.
2. Neutron stars quasinormal modes can resonantly excited in stars that are in a binary system when the orbital frequency of the binary is equal or close to the oscillation frequency of the mode (remember that the orbital period of the binary changes because of the emission of gravitational waves). Using typical values for the neutron stars' masses and radii, estimate the maximum orbital frequency of the binary and conclude which modes can be resonantly excited through this mechanism.
3. Suppose that a future gravitational wave observation with a very sensitive detector was able to detect an f-mode with a frequency $f = 1.60 \pm 0.08$ kHz and a damping time $\tau = 0.30 \pm 0.09$ s. Plot the universal relations obtained for a sample of different EOSs and use the observations to determine the mass and radius of the star, with their corresponding uncertainties.
4. What happens if we try to calculate the f-modes for a neutron star in the unstable branch of the mass radius relation? Stars in the unstable branch are unstable against radial oscillations, but the f-mode is a non-radial mode of oscillation. Try to use the universal relations to estimate where (or if) the f-mode should become unstable. How does that compare with the mass-radius relation?
5. Consider a magnetized neutron star. The magnetic energy can be estimated as $E_B \sim B^2 R^3$ and the gravitational energy can be estimated as $E_g \sim GM^2/R$. If the magnetic field is too strong, it will start distorting the spherical symmetry of the star. Calculate the ratio E_B/E_g to estimate how strong the magnetic field must be to be able to distort the star.

6. **Computer project:** This is a longer project to be developed over the next 2 weeks, with the main goal to implement numerically the full equations derived by Lindblom & Detweiler to calculate the f-mode frequencies and damping times of a star. Follow steps given in the appendices of Lindblom & Detweiler (1983) and use the final form of the equations given by Detweiler & Lindblom (1985).

Neutron Star Asteroseismology

Problem Set 4

22/05/2019

1. We can use the period of rotation of the fastest known pulsar (1.56 ms) to estimate the radius of a millisecond pulsar. For the star to hold together at its equator under the opposing gravitational and centrifugal forces we must have

$$\zeta^2 \frac{GmM}{R^2} > m\Omega^2 R\,. \tag{1}$$

   The factor ζ is unity in Newtonian physics and is found empirically to be about 0.65 in General Relativity. Use this information and a reliable estimate for the maximum mass of a neutron star and find how large its radius can be for the star to avoid mass shedding.

2. Use an equality in eq. (1) to find an estimate for Ω_K. Show that a rotating neutron star will have

$$\frac{E_{\rm rot}}{E_{\rm grav}} \approx 0.13 \left(\frac{\Omega}{\Omega_K}\right)^2 . \tag{2}$$

3. Look up the rotational correction to the f-mode frequency given by Ferrari, Gualtieri and Marassi (2007) (see eq. (59) and Fig. 1 (b)). Find how fast a neutron star must be spinning for the f-mode to be resonantly excited in a circular neutron star binary system.

4. Find the velocity of a particle in a circular stable orbit at the equator of a rotating star. Note that it is different according to whether the particle is co-rotating or counter-rotating.

5. Computer problem: Use the estimates given for τ_{GW}, τ_{BV} and τSV collected by Haskell, Degenaar and Ho (2012) (see eqs. (3), (5) and (6)) to plot the r-mode instability window by solving

$$\frac{1}{\tau_{GW}} + \frac{1}{\tau_{BV}} + \frac{1}{\tau_{SV}} = 0\,, \tag{3}$$

   as in their Fig. 1. Add to your plot the stars listed on their table 1. Which stars are inside the window?