

NONTRIVIAL TWISTED SUMS OF c_0 AND $C(K)$

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ABSTRACT. We obtain a new class of compact Hausdorff spaces K for which c_0 can be nontrivially twisted with $C(K)$.

1. INTRODUCTION

In this article, we present a broad new class of compact Hausdorff spaces K such that there exists a nontrivial twisted sum of c_0 and $C(K)$, where $C(K)$ denotes the Banach space of continuous real-valued functions on K endowed with the supremum norm. By a *twisted sum* of the Banach spaces Y and X we mean a short exact sequence $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$, where Z is a Banach space and the maps are bounded linear operators. This twisted sum is called *trivial* if the exact sequence splits, i.e., if the map $Y \rightarrow Z$ admits a bounded linear left inverse (equivalently, if the map $Z \rightarrow X$ admits a bounded linear right inverse). In other words, the twisted sum is trivial if the range of the map $Y \rightarrow Z$ is complemented in Z ; in this case, $Z \cong X \oplus Y$. As in [7], we denote by $\text{Ext}(X, Y)$ the set of equivalence classes of twisted sums of Y and X and we write $\text{Ext}(X, Y) = 0$ if every such twisted sum is trivial.

Many problems in Banach space theory are related to the quest for conditions under which $\text{Ext}(X, Y) = 0$. For instance, an equivalent statement for the classical Theorem of Sobczyk ([5, 13]) is that if X is a separable Banach space, then $\text{Ext}(X, c_0) = 0$ ([3, Proposition 3.2]). The converse of the latter statement clearly does not hold in general: for example, $\text{Ext}(\ell_1(I), c_0) = 0$, since $\ell_1(I)$ is a projective Banach space. However, the following question remains open: *is it true that $\text{Ext}(C(K), c_0) \neq 0$ for any nonseparable $C(K)$ space?* This problem was stated in [4, 5] and further studied in the recent article [6], in which the author proves that, under the *continuum hypothesis* (CH), the space $\text{Ext}(C(K), c_0)$ is nonzero for a nonmetrizable compact Hausdorff space K of finite height. In addition to this result, everything else that is known about the problem is summarized in [6, Proposition 2], namely that $\text{Ext}(C(K), c_0)$ is nonzero for a $C(K)$ space under any one of the following assumptions:

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- K is a nonmetrizable Eberlein compact space;
- K is a Valdivia compact space which does not satisfy the countable chain condition (ccc);
- the weight of K is equal to ω_1 and the dual space of $C(K)$ is not weak*-separable;
- K has the extension property ([8]) and it does not have ccc;
- $C(K)$ contains an isomorphic copy of ℓ_∞ .

Note also that if $\text{Ext}(Y, c_0) \neq 0$ and X contains a *complemented* isomorphic copy of Y , then $\text{Ext}(X, c_0) \neq 0$.

Here is an overview of the main results of this article. Theorem 2.3 gives a condition involving biorthogonal systems in a Banach space X which implies that $\text{Ext}(X, c_0) \neq 0$. In the rest of Section 2, we discuss some of its implications when X is of the form $C(K)$. It is proven that if K contains a homeomorphic copy of $[0, \omega] \times [0, \mathfrak{c}]$ or of $2^{\mathfrak{c}}$, then $\text{Ext}(C(K), c_0)$ is nonzero, where $\mathfrak{c}$ denotes the cardinality of the *continuum*. In Sections 3 and 4, we investigate the consequences of the results of Section 2 for Valdivia and Corson compacta. Recall that Valdivia compact spaces constitute a large superclass of Corson compact spaces closed under arbitrary products; moreover, every Eberlein compact is a Corson compact (see [11] for a survey on Valdivia compacta). Section 3 is devoted to the proof that, under CH, it holds that $\text{Ext}(C(K), c_0) \neq 0$ for every nonmetrizable Corson compact space K . The question of whether $\text{Ext}(C(K), c_0) \neq 0$ for an arbitrary nonmetrizable Valdivia compact space K remains open (even under CH), but in Section 4 we solve some particular cases of this problem.

2. GENERAL RESULTS

Throughout the paper, the weight and the density character of a topological space $\mathcal{X}$ are denoted, respectively, by $w(\mathcal{X})$ and $\text{dens}(\mathcal{X})$. Moreover, we always denote by χ_A the characteristic function of a set A and by $|A|$ the cardinality of A . We start with a technical lemma which is the heart of the proof of Theorem 2.3. A family of sets $(A_i)_{i \in I}$ is said to be *almost disjoint* if each A_i is infinite and $A_i \cap A_j$ is finite, for all $i, j \in I$ with $i \neq j$.

Lemma 2.1. *There exists an almost disjoint family $(A_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ of subsets of ω satisfying the following property: for every family $(A'_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ with each $A'_{n,\alpha} \subset A_{n,\alpha}$ cofinite in $A_{n,\alpha}$, it holds that $\sup_{p \in \omega} |M_p| = +\infty$, where:*

$$M_p = \{n \in \omega : p \in \bigcup_{\alpha \in \mathfrak{c}} A'_{n,\alpha}\}.$$

Proof. We will obtain an almost disjoint family $(A_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ of subsets of $2^{<\omega}$ with the desired property, where $2^{<\omega} = \bigcup_{k \in \omega} 2^k$ denotes the set of finite sequences in $2 = \{0, 1\}$. For each $\epsilon \in 2^\omega$, we set:

$$\mathcal{A}_\epsilon = \{\epsilon|_k : k \in \omega\},$$

so that $(\mathcal{A}_\epsilon)_{\epsilon \in 2^\omega}$ is an almost disjoint family of subsets of $2^{<\omega}$. Let $(\mathcal{B}_\alpha)_{\alpha \in \mathfrak{c}}$ be an enumeration of the uncountable Borel subsets of 2^ω . Recalling that

$|\mathcal{B}_\alpha| = \mathfrak{c}$ for all $\alpha \in \mathfrak{c}$ ([12, Theorem 13.6]), one easily obtains by transfinite recursion a family $(\epsilon_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ of pairwise distinct elements of 2^ω such that $\epsilon_{n,\alpha} \in \mathcal{B}_\alpha$, for all $n \in \omega$, $\alpha \in \mathfrak{c}$. Set $A_{n,\alpha} = \mathcal{A}_{\epsilon_{n,\alpha}}$ and let $(A'_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ be as in the statement of the lemma. For $n \in \omega$, denote by D_n the set of those $\epsilon \in 2^\omega$ such that $n \in M_p$ for all but finitely many $p \in \mathcal{A}_\epsilon$. Note that:

$$D_n = \bigcup_{k_0 \in \omega} \bigcap_{k \geq k_0} \bigcup \{C_p : p \in 2^k \text{ with } n \in M_p\},$$

where C_p denotes the clopen subset of 2^ω consisting of the extensions of p . The above equality implies that D_n is an F_σ (and, in particular, a Borel) subset of 2^ω . We claim that the complement of D_n in 2^ω is countable. Namely, if it were uncountable, there would exist $\alpha \in \mathfrak{c}$ with $\mathcal{B}_\alpha = 2^\omega \setminus D_n$. But, since $n \in M_p$ for all $p \in A'_{n,\alpha}$, we have that $\epsilon_{n,\alpha} \in D_n$, contradicting the fact that $\epsilon_{n,\alpha} \in \mathcal{B}_\alpha$ and proving the claim. To conclude the proof of the lemma, note that for each $n \geq 1$ the intersection $\bigcap_{i < n} D_i$ is nonempty; for $\epsilon \in \bigcap_{i < n} D_i$, we have that $\{i : i < n\} \subset M_p$, for all but finitely many $p \in \mathcal{A}_\epsilon$. $\square$

Let X be a Banach space. Recall that a *biorthogonal system* in X is a family $(x_i, \gamma_i)_{i \in I}$ with $x_i \in X$, $\gamma_i \in X^*$, $\gamma_i(x_i) = 1$ and $\gamma_i(x_j) = 0$ for $i \neq j$. The *cardinality* of the biorthogonal system $(x_i, \gamma_i)_{i \in I}$ is defined as the cardinality of I .

Definition 2.2. Let $(x_i, \gamma_i)_{i \in I}$ be a biorthogonal system in a Banach space X . We call $(x_i, \gamma_i)_{i \in I}$ *bounded* if $\sup_{i \in I} \|x_i\| < +\infty$ and $\sup_{i \in I} \|\gamma_i\| < +\infty$; *weak*-null* if $(\gamma_i)_{i \in I}$ is a weak*-null family, i.e., if $(\gamma_i(x))_{i \in I}$ is in $c_0(I)$, for all $x \in X$.

Theorem 2.3. Let X be a Banach space. Assume that there exist a weak*-null biorthogonal system $(x_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ in X and a constant $C \geq 0$ such that:

$$\left\| \sum_{i=1}^k x_{n_i, \alpha_i} \right\| \leq C,$$

for all $n_1, \dots, n_k \in \omega$ pairwise distinct, all $\alpha_1, \dots, \alpha_k \in \mathfrak{c}$, and all $k \geq 1$. Then $\text{Ext}(X, c_0) \neq 0$.

Proof. By [7, Proposition 1.4.f], we have that $\text{Ext}(X, c_0) = 0$ if and only if every bounded operator $T : X \rightarrow \ell_\infty/c_0$ admits a *lifting*¹, i.e., a bounded operator $\widehat{T} : X \rightarrow \ell_\infty$ with $T(x) = \widehat{T}(x) + c_0$, for all $x \in X$. Let us then show that there exists an operator $T : X \rightarrow \ell_\infty/c_0$ that does not admit a lifting. To this aim, let $(A_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ be an almost disjoint family as in Lemma 2.1 and consider the unique isometric embedding $S : c_0(\omega \times \mathfrak{c}) \rightarrow \ell_\infty/c_0$ such that $S(e_{n,\alpha}) = \chi_{A_{n,\alpha}} + c_0$, where $(e_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ denotes the canonical basis of

¹More concretely, a nontrivial twisted sum of c_0 and X is obtained by considering the pull-back of the short exact sequence $0 \rightarrow c_0 \rightarrow \ell_\infty \rightarrow \ell_\infty/c_0 \rightarrow 0$ by an operator $T : X \rightarrow \ell_\infty/c_0$ that does not admit a lifting.

$c_0(\omega \times \mathfrak{c})$. Denote by $\Gamma : X \rightarrow c_0(\omega \times \mathfrak{c})$ the bounded operator with coordinate functionals $(\gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ and set $T = S \circ \Gamma : X \rightarrow \ell_\infty/c_0$. Assuming by contradiction that there exists a lifting $\widehat{T}$ of T and denoting by $(\mu_p)_{p \in \omega}$ the sequence of coordinate functionals of $\widehat{T}$, we have that the set:

$$A'_{n,\alpha} = \{p \in A_{n,\alpha} : \mu_p(x_{n,\alpha}) \geq \frac{1}{2}\}$$

is cofinite in $A_{n,\alpha}$. It follows that for each $k \geq 1$, there exist $p \in \omega$, $n_1, \dots, n_k \in \omega$ pairwise distinct, and $\alpha_1, \dots, \alpha_k \in \mathfrak{c}$ such that $p \in A'_{n_i, \alpha_i}$, for $i = 1, \dots, k$. Hence:

$$\frac{k}{2} \leq \mu_p\left(\sum_{i=1}^k x_{n_i, \alpha_i}\right) \leq C\|\widehat{T}\|,$$

which yields a contradiction. $\square$

Corollary 2.4. *Let K be a compact Hausdorff space. Assume that there exists a bounded weak*-null biorthogonal system $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ in $C(K)$ such that $f_{n,\alpha} f_{m,\beta} = 0$, for all $n, m \in \omega$ with $n \neq m$ and all $\alpha, \beta \in \mathfrak{c}$. Then $\text{Ext}(C(K), c_0) \neq 0$.* $\square$

Definition 2.5. We say that a compact Hausdorff space K satisfies *property (*)* if there exist a sequence $(F_n)_{n \in \omega}$ of closed subsets of K and a bounded weak*-null biorthogonal system $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ in $C(K)$ such that:

$$(1) \quad F_n \cap \overline{\bigcup_{m \neq n} F_m} = \emptyset$$

and $\text{supp } f_{n,\alpha} \subset F_n$, for all $n \in \omega$ and all $\alpha \in \mathfrak{c}$, where $\text{supp } f_{n,\alpha}$ denotes the support of $f_{n,\alpha}$.

In what follows, we denote by $M(K)$ the space of finite countably-additive signed regular Borel measures on K , endowed with the total variation norm. We identify as usual the dual space of $C(K)$ with $M(K)$.

Lemma 2.6. *Let K be a compact Hausdorff space and L be a closed subspace of K . If L satisfies property (*), then so does K .*

Proof. Consider, as in Definition 2.5, a sequence $(F_n)_{n \in \omega}$ of closed subsets of L and a bounded weak*-null biorthogonal system $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ in $C(L)$. By recursion on n , one easily obtains a sequence $(U_n)_{n \in \omega}$ of pairwise disjoint open subsets of K with each U_n containing F_n . Let V_n be an open subset of K with $F_n \subset V_n \subset \overline{V_n} \subset U_n$. Using Tietze's Extension Theorem and Urysohn's Lemma, we get a continuous extension $\tilde{f}_{n,\alpha}$ of $f_{n,\alpha}$ to K with support contained in $\overline{V_n}$ and having the same norm as $f_{n,\alpha}$. To conclude the proof, let $\tilde{\gamma}_{n,\alpha} \in M(K)$ be the extension of $\gamma_{n,\alpha} \in M(L)$ that vanishes identically outside of L and observe that $(\tilde{f}_{n,\alpha}, \tilde{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ is a bounded weak*-null biorthogonal system in $C(K)$. $\square$

As an immediate consequence of Lemma 2.6 and Corollary 2.4, we obtain the following result.

Theorem 2.7. *If a compact Hausdorff space L satisfies property $(*)$, then every compact Hausdorff space K containing a homeomorphic copy of L satisfies $\text{Ext}(C(K), c_0) \neq 0$. $\square$*

We now establish a few results which give sufficient conditions for a space K to satisfy property $(*)$. Recall that, given a closed subset F of a compact Hausdorff space K , an *extension operator* for F in K is a bounded operator $E : C(F) \rightarrow C(K)$ which is a right inverse for the restriction operator $C(K) \ni f \mapsto f|_F \in C(F)$. Note that F admits an extension operator in K if and only if the kernel

$$C(K|F) = \{f \in C(K) : f|_F = 0\}$$

of the restriction operator is complemented in $C(K)$. A point x of a topological space $\mathcal{X}$ is called a *cluster point* of a sequence $(S_n)_{n \in \omega}$ of subsets of $\mathcal{X}$ if every neighborhood of x intersects S_n for infinitely many $n \in \omega$.

Lemma 2.8. *Let K be a compact Hausdorff space. Assume that there exist a sequence $(F_n)_{n \in \omega}$ of pairwise disjoint closed subsets of K and a closed subset F of K satisfying the following conditions:*

- (a) *F admits an extension operator in K ;*
- (b) *every cluster point of $(F_n)_{n \in \omega}$ is in F and $F_n \cap F = \emptyset$, for all $n \in \omega$;*
- (c) *there exists a family $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$, where $(f_{n,\alpha}, \gamma_{n,\alpha})_{\alpha \in \mathfrak{c}}$ is a weak*-null biorthogonal system in $C(F_n)$ for each $n \in \omega$ and*

$$\sup_{n \in \omega, \alpha \in \mathfrak{c}} \|f_{n,\alpha}\| < +\infty, \quad \sup_{n \in \omega, \alpha \in \mathfrak{c}} \|\gamma_{n,\alpha}\| < +\infty.$$

Then K satisfies property $()$.*

Proof. From (b) and the fact that the F_n are pairwise disjoint it follows that (1) holds. Let $(U_n)_{n \in \omega}$, $(V_n)_{n \in \omega}$, and $(\tilde{f}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ be as in the proof of Lemma 2.6; we assume also that $\bar{V}_n \cap F = \emptyset$, for all $n \in \omega$. Let $\tilde{\gamma}_{n,\alpha} \in M(K)$ be the extension of $\gamma_{n,\alpha} \in M(F_n)$ that vanishes identically outside of F_n . We have that $(\tilde{f}_{n,\alpha}, \tilde{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ is a bounded biorthogonal system in $C(K)$ and that $(\tilde{\gamma}_{n,\alpha})_{\alpha \in \mathfrak{c}}$ is weak*-null for each n , though it is not true in general that the entire family $(\tilde{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ is weak*-null. In order to take care of this problem, let $P : C(K) \rightarrow C(K|F)$ be a bounded projection and set $\hat{\gamma}_{n,\alpha} = \tilde{\gamma}_{n,\alpha} \circ P$. Since all $\tilde{f}_{n,\alpha}$ are in $C(K|F)$, we have that $(\tilde{f}_{n,\alpha}, \hat{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ is biorthogonal. To prove that $(\hat{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ is weak*-null, note that (b) implies that $\lim_{n \rightarrow +\infty} \|f|_{F_n}\| = 0$, for all $f \in C(K|F)$. $\square$

Corollary 2.9. *Let K be a compact Hausdorff space. If $C(K)$ admits a bounded weak*-null biorthogonal system of cardinality $\mathfrak{c}$, then the space $[0, \omega] \times K$ satisfies property $(*)$. In particular, $L \times K$ satisfies property $(*)$ for every compact Hausdorff space L containing a nontrivial convergent sequence.*

Proof. Take $F_n = \{n\} \times K$, $F = \{\omega\} \times K$, and use the fact that F is a retract of $[0, \omega] \times K$ and thus admits an extension operator in $[0, \omega] \times K$. $\square$

Corollary 2.10. *The spaces $[0, \omega] \times [0, \mathfrak{c}]$ and $2^{\mathfrak{c}}$ satisfy property (*). In particular, a product of at least $\mathfrak{c}$ compact Hausdorff spaces with more than one point satisfies property (*).*

Proof. The family $(\chi_{[0, \alpha]}, \delta_\alpha - \delta_{\alpha+1})_{\alpha \in \mathfrak{c}}$ is a bounded weak*-null biorthogonal system in $C([0, \mathfrak{c}])$, where $\delta_\alpha \in M([0, \mathfrak{c}])$ denotes the probability measure with support $\{\alpha\}$. It follows from Corollary 2.9 that $[0, \omega] \times [0, \mathfrak{c}]$ satisfies property (*). To see that $2^{\mathfrak{c}}$ also does, note that the map $[0, \mathfrak{c}] \ni \alpha \mapsto \chi_\alpha \in 2^{\mathfrak{c}}$ embeds $[0, \mathfrak{c}]$ into $2^{\mathfrak{c}}$, so that $2^{\mathfrak{c}} \cong 2^\omega \times 2^{\mathfrak{c}}$ contains a homeomorphic copy of $[0, \omega] \times [0, \mathfrak{c}]$. $\square$

Recall that a *projectional resolution of the identity* (PRI) of a Banach space X is a family $(P_\alpha)_{\omega \leq \alpha \leq \text{dens}(X)}$ of projection operators $P_\alpha : X \rightarrow X$ satisfying the following conditions:

- $\|P_\alpha\| = 1$, for $\omega \leq \alpha \leq \text{dens}(X)$;
- $P_{\text{dens}(X)}$ is the identity of X ;
- $P_\alpha[X] \subset P_\beta[X]$ and $\text{Ker}(P_\beta) \subset \text{Ker}(P_\alpha)$, for $\omega \leq \alpha \leq \beta \leq \text{dens}(X)$;
- $P_\alpha(x) = \lim_{\beta < \alpha} P_\beta(x)$, for all $x \in X$, if $\omega < \alpha \leq \text{dens}(X)$ is a limit ordinal;
- $\text{dens}(P_\alpha[X]) \leq |\alpha|$, for $\omega \leq \alpha \leq \text{dens}(X)$.

We call the PRI *strictly increasing* if $P_\alpha[X]$ is a proper subspace of $P_\beta[X]$, for $\omega \leq \alpha < \beta \leq \text{dens}(X)$.

Corollary 2.11. *Let K and L be compact Hausdorff spaces such that L contains a nontrivial convergent sequence and $w(K) \geq \mathfrak{c}$. If $C(K)$ admits a strictly increasing PRI, then the space $L \times K$ satisfies property (*).*

Proof. This follows from Corollary 2.9 by observing that if a Banach space X admits a strictly increasing PRI, then X admits a weak*-null biorthogonal system $(x_\alpha, \gamma_\alpha)_{\omega \leq \alpha < \text{dens}(X)}$ with $\|x_\alpha\| = 1$ and $\|\gamma_\alpha\| \leq 2$, for all α . Namely, pick a unit vector x_α in $P_{\alpha+1}[X] \cap \text{Ker}(P_\alpha)$ and set $\gamma_\alpha = \phi_\alpha \circ (P_{\alpha+1} - P_\alpha)$, where $\phi_\alpha \in X^*$ is a norm-one functional satisfying $\phi_\alpha(x_\alpha) = 1$. $\square$

3. NONTRIVIAL TWISTED SUMS FOR CORSON COMPACTA

Let us recall some standard definitions. Given an index set I , we write $\Sigma(I) = \{x \in \mathbb{R}^I : \text{supp } x \text{ is countable}\}$, where the support $\text{supp } x$ of x is defined by $\text{supp } x = \{i \in I : x_i \neq 0\}$. Given a compact Hausdorff space K , we call A a Σ -subset of K if there exist an index set I and a continuous injection $\varphi : K \rightarrow \mathbb{R}^I$ such that $A = \varphi^{-1}[\Sigma(I)]$. The space K is called a *Valdivia compactum* if it admits a dense Σ -subset and it is called a *Corson compactum* if K is a Σ -subset of itself. This section will be dedicated to the proof of the following result.

Theorem 3.1. *If K is a Corson compact space with $w(K) \geq \mathfrak{c}$, then $\text{Ext}(C(K), c_0) \neq 0$. In particular, under CH, we have $\text{Ext}(C(K), c_0) \neq 0$ for any nonmetrizable Corson compact space K .*

The fact that $\text{Ext}(C(K), c_0) \neq 0$ for a Valdivia compact space K which does not have ccc is already known ([6, Proposition 2]). Our strategy for the proof of Theorem 3.1 is to use Lemma 2.8 to show that if K is a Corson compact space with $w(K) \geq \mathfrak{c}$ having ccc, then K satisfies property (*). We start with a lemma that will be used as a tool for verifying the assumptions of Lemma 2.8. Recall that a closed subset of a topological space is called *regular* if it is the closure of an open set (equivalently, if it is the closure of its own interior). Obviously, a closed subset of a Corson compact space is again Corson and a regular closed subset of a Valdivia compact space is again Valdivia.

Lemma 3.2. *Let K be a compact Hausdorff space and F be a closed non-open G_δ subset of K . Then there exists a sequence $(F_n)_{n \in \omega}$ of nonempty pairwise disjoint regular closed subsets of K such that condition (b) in the statement of Lemma 2.8 holds.*

Proof. We can write $F = \bigcap_{n \in \omega} V_n$, with each V_n open in K and $\overline{V_{n+1}}$ properly contained in V_n . Set $U_n = V_n \setminus \overline{V_{n+1}}$, so that all cluster points of $(U_n)_{n \in \omega}$ are in F . To conclude the proof, let F_n be a nonempty regular closed set contained in U_n . $\square$

Once we get the closed sets $(F_n)_{n \in \omega}$ from Lemma 3.2, we still have to verify the rest of the conditions in the statement of Lemma 2.8. First, we need an assumption ensuring that $w(F_n) \geq \mathfrak{c}$, for all n . To this aim, given a point x of a topological space $\mathcal{X}$, we define the *weight of x in $\mathcal{X}$* by:

$$w(x, \mathcal{X}) = \min \{w(V) : V \text{ neighborhood of } x \text{ in } \mathcal{X}\}.$$

Recall that if K is a Valdivia compact space, then $C(K)$ admits a PRI ([15, Theorem 2]). Moreover, a trivial adaptation of the proof in [15] shows in fact that $C(K)$ admits a strictly increasing PRI. Thus, by the argument in the proof of Corollary 2.11, $C(K)$ admits a weak*-null biorthogonal system $(f_\alpha, \gamma_\alpha)_{\omega \leq \alpha < w(K)}$ such that $\|f_\alpha\| \leq 1$ and $\|\gamma_\alpha\| \leq 2$, for all α . The following result is now immediately obtained.

Corollary 3.3. *Let K be a Valdivia compact space such that $w(x, K) \geq \mathfrak{c}$, for all $x \in K$. Assume that there exists a closed non-open G_δ subset F admitting an extension operator in K . Then K satisfies property (*). $\square$*

Assuming that K has ccc, the next lemma allows us to reduce the proof of Theorem 3.1 to the case when $w(x, K) \geq \mathfrak{c}$, for all $x \in K$.

Lemma 3.4. *Let K be a ccc Valdivia compact space and set:*

$$H = \{x \in K : w(x, K) \geq \mathfrak{c}\}.$$

Then:

- (a) $H \neq \emptyset$, if $w(K) \geq \mathfrak{c}$;
- (b) $w(K \setminus \text{int}(H)) < \mathfrak{c}$, where $\text{int}(H)$ denotes the interior of H ;
- (c) H is a regular closed subset of K ;

(d) $w(x, H) \geq \mathfrak{c}$, for all $x \in H$.

Proof. If $H = \emptyset$, then K can be covered by a finite number of open sets with weight less than $\mathfrak{c}$, so that $w(K) < \mathfrak{c}$. This proves (a). To prove (b), let $(U_i)_{i \in I}$ be maximal among antichains of open subsets of K with weight less than $\mathfrak{c}$. Since I is countable and $\mathfrak{c}$ has uncountable cofinality, we have that $U = \bigcup_{i \in I} U_i$ has weight less than $\mathfrak{c}$. From the maximality of $(U_i)_{i \in I}$, it follows that $K \setminus H \subset \overline{U}$; then $K \setminus \text{int}(H) = \overline{K \setminus H} \subset \overline{U}$. To conclude the proof of (b), let us show that $w(\overline{U}) < \mathfrak{c}$. Let A be a dense Σ -subset of K and let D be a dense subset of $A \cap U$ with $|D| \leq w(U)$. Then $\overline{D}$ is homeomorphic to a subspace of $\mathbb{R}^{w(U)}$, so that $w(\overline{U}) = w(\overline{D}) \leq w(U) < \mathfrak{c}$. To prove (c), note that H is clearly closed; moreover, by (b), the open set $K \setminus \text{int}(H)$ has weight less than $\mathfrak{c}$ and hence it is contained in $K \setminus H$. Finally, to prove (d), let V be a closed neighborhood in K of some $x \in H$. By (b), we have $w(V \setminus H) < \mathfrak{c}$. Recall from [9, p. 26] that if a compact Hausdorff space is the union of not more than κ subsets of weight not greater than κ , then the weight of the space is not greater than κ . Since $w(V) \geq \mathfrak{c}$, it follows from such result that $w(V \cap H) \geq \mathfrak{c}$. $\square$

Proof of Theorem 3.1. By Lemma 3.4, it suffices to prove that if K is a nonempty Corson compact space such that $w(x, K) \geq \mathfrak{c}$ for all $x \in K$, then K satisfies property (*). Since a nonempty Corson compact space K admits a G_δ point x ([11, Theorem 3.3]), this fact follows from Corollary 3.3 with $F = \{x\}$. $\square$

Remark 3.5. It is known that under Martin's Axiom (MA) and the negation of CH, every ccc Corson compact is metrizable ([1]). Thus, Theorem 3.1 implies that $\text{Ext}(C(K), c_0) \neq 0$ for every nonmetrizable Corson compact space K under MA.

4. TOWARDS THE GENERAL VALDIVIA CASE

In this section we prove that $\text{Ext}(C(K), c_0) \neq 0$ for certain classes of nonmetrizable Valdivia compact spaces K and we propose a strategy for dealing with the general problem. First, let us state some results which are immediate consequences of what we have done so far.

Proposition 4.1. *If K is a Valdivia compact space with $w(K) \geq \mathfrak{c}$ and L is a compact Hausdorff space containing a nontrivial convergent sequence, then $L \times K$ satisfies property (*).*

Proof. As we have observed in Section 3, if K is a Valdivia compact space, then $C(K)$ admits a strictly increasing PRI. The conclusion follows from Corollary 2.11. $\square$

Proposition 4.2. *Let K be a Valdivia compact space admitting a G_δ point x with $w(x, K) \geq \mathfrak{c}$. Then $\text{Ext}(C(K), c_0) \neq 0$ and, if K has ccc, then K satisfies property (*).*

Proof. As mentioned before, the non-ccc case is already known. Assuming that K has ccc, define H as in Lemma 3.4 and conclude that H satisfies property $(*)$ using Corollary 3.3 with $F = \{x\}$. $\square$

Corollary 4.3. *Let K be a Valdivia compact space with $w(K) \geq \mathfrak{c}$ admitting a dense Σ -subset A such that $K \setminus A$ is of first category. Then $\text{Ext}(C(K), c_0) \neq 0$ and, if K has ccc, then K satisfies property $(*)$.*

Proof. By [11, Theorem 3.3], K has a dense subset of G_δ points. Assuming that K has ccc and defining H as in Lemma 3.4, we obtain that H contains a G_δ point of K , which implies that K satisfies the assumptions of Proposition 4.2. $\square$

Now we investigate conditions under which a Valdivia compact space K contains a homeomorphic copy of $[0, \omega] \times [0, \mathfrak{c}]$. Given an index set I and a subset J of I , we denote by $r_J : \mathbb{R}^I \rightarrow \mathbb{R}^I$ the map defined by setting $r_J(x)|_J = x|_J$ and $r_J(x)|_{I \setminus J} \equiv 0$, for all $x \in \mathbb{R}^I$. Following [2], given a subset K of $\mathbb{R}^I$, we say that $J \subset I$ is K -good if $r_J[K] \subset K$. In [2, Lemma 1.2], it is proven that if K is a compact subset of $\mathbb{R}^I$ and $\Sigma(I) \cap K$ is dense in K , then every infinite subset J of I is contained in a K -good set J' with $|J| = |J'|$.

Proposition 4.4. *Let K be a Valdivia compact space admitting a dense Σ -subset A such that some point of $K \setminus A$ is the limit of a nontrivial sequence in K . Then K contains a homeomorphic copy of $[0, \omega] \times [0, \omega_1]$. In particular, assuming CH , we have that K satisfies property $(*)$.*

Proof. We can obviously assume that K is a compact subset of some $\mathbb{R}^I$ and that $A = \Sigma(I) \cap K$. Since A is sequentially closed, our hypothesis implies that there exists a continuous injective map $[0, \omega] \ni n \mapsto x_n \in K \setminus A$. Let J be a countable subset of I such that $x_n|_J \neq x_m|_J$, for all $n, m \in [0, \omega]$ with $n \neq m$. Using [2, Lemma 1.2] and transfinite recursion, one easily obtains a family $(J_\alpha)_{\alpha \leq \omega_1}$ of K -good subsets of I satisfying the following conditions:

- J_α is countable, for $\alpha < \omega_1$;
- $J \subset J_0$;
- $J_\alpha \subset J_\beta$, for $0 \leq \alpha \leq \beta \leq \omega_1$;
- $J_\alpha = \bigcup_{\beta < \alpha} J_\beta$, for limit $\alpha \in [0, \omega_1]$;
- for all $n \in [0, \omega]$, the map $[0, \omega_1] \ni \alpha \mapsto J_\alpha \cap \text{supp } x_n$ is injective.

Given these conditions, it is readily checked that the map

$$[0, \omega] \times [0, \omega_1] \ni (n, \alpha) \longmapsto r_{J_\alpha}(x_n) \in K$$

is continuous and injective. $\square$

Remark 4.5. The following converse of Proposition 4.4 also holds: if K is a Valdivia compact space containing a homeomorphic copy of $[0, \omega] \times [0, \omega_1]$, then $K \setminus A$ contains a nontrivial convergent sequence, for *any* dense Σ -subset A of K . Namely, if $\phi : [0, \omega] \times [0, \omega_1] \rightarrow K$ is a continuous injection, then $\phi(\omega, \alpha) \in K \setminus A$ for some $\alpha \in [0, \omega_1]$, since $[0, \omega_1]$ is not Corson ([11, Example 1.10 (i)]). Moreover, the nontrivial sequence $(\phi(n, \alpha))_{n \in \omega}$ converges

to $\phi(\omega, \alpha)$. One consequence of this observation is that if $K \setminus A$ contains a nontrivial convergent sequence for *some* dense Σ -subset A of K , then $K \setminus A$ contains a nontrivial convergent sequence for *any* dense Σ -subset A of K .

Remark 4.6. If a Valdivia compact space K admits two distinct dense Σ -subsets, then the assumption of Proposition 4.4 holds for K . Namely, given dense Σ -subsets A and B of K and a point $x \in A \setminus B$, then x is not isolated, since B is dense. Moreover, x is not isolated in A , because A is dense. Finally, since A is a Fréchet–Urysohn space ([11, Lemma 1.6 (ii)]), x is the limit of a sequence in $A \setminus \{x\}$.

Finally, we observe that the validity of the following conjecture would imply, under CH, that $\text{Ext}(C(K), c_0) \neq 0$ for any nonmetrizable Valdivia compact space K .

Conjecture. *If K is a nonempty Valdivia compact space having ccc, then either K has a G_δ point or K admits a nontrivial convergent sequence in the complement of a dense Σ -subset.*

To see that the conjecture implies the desired result, use Lemma 3.4 and Propositions 4.2 and 4.4, keeping in mind that a regular closed subset of a ccc space has ccc as well. The conjecture remains open, but in what follows we present an example showing that it is false if the assumption that K has ccc is removed.

Recall that a *tree* is a partially ordered set $(T, \leq)$ such that, for all $t \in T$, the set $(\cdot, t) = \{s \in T : s < t\}$ is well-ordered. As in [14, p. 288], we define a compact Hausdorff space from a tree T by considering the subspace $P(T)$ of 2^T consisting of all characteristic functions of paths of T ; by a *path* of T we mean a totally ordered subset A of T such that $(\cdot, t) \subset A$, for all $t \in A$. It is easy to see that $P(T)$ is closed in 2^T ; we call it the *path space* of T .

Denote by $S(\omega_1)$ the set of countable successor ordinals and consider the tree $T = \bigcup_{\alpha \in S(\omega_1)} \omega_1^\alpha$, partially ordered by inclusion. The path space $P(T)$ is the image of the injective map $\Lambda \ni \lambda \mapsto \chi_{A(\lambda)} \in 2^T$, where $\Lambda = \bigcup_{\alpha \leq \omega_1} \omega_1^\alpha$ and $A(\lambda) = \{t \in T : t \subset \lambda\}$.

Proposition 4.7. *If the tree T is defined as above, then its path space $P(T)$ is a compact subspace of $\mathbb{R}^T$ satisfying the following conditions:*

- (a) $P(T) \cap \Sigma(T)$ is dense in $P(T)$, so that $P(T)$ is Valdivia;
- (b) $P(T)$ has no G_δ points;
- (c) no point of $P(T) \setminus \Sigma(T)$ is the limit of a nontrivial sequence in $P(T)$.

Proof. To prove (a), note that $\chi_{A(\lambda)} = \lim_{\alpha < \omega_1} \chi_{A(\lambda|_\alpha)}$ for all $\lambda \in \omega_1^{\omega_1}$. Let us prove (b). Since $P(T)$ is Valdivia, every G_δ point of $P(T)$ must be in $\Sigma(T)$ ([10, Proposition 2.2 (3)]), i.e., it must be of the form $\chi_{A(\lambda)}$, with $\lambda \in \omega_1^\alpha$, $\alpha < \omega_1$. To see that $\chi_{A(\lambda)}$ cannot be a G_δ point of $P(T)$, it suffices to check that for any countable subset E of T , there exists $\mu \in \Lambda$, $\mu \neq \lambda$, such that $\chi_{A(\lambda)}$ and $\chi_{A(\mu)}$ are identical on E . To this aim, simply take

$\mu = \lambda \cup \{(\alpha, \beta)\}$, with $\beta \in \omega_1 \setminus \{t(\alpha) : t \in E \text{ and } \alpha \in \text{dom}(t)\}$. Finally, to prove (c), let $(\chi_{A(\lambda_n)})_{n \geq 1}$ be a sequence of pairwise distinct elements of $P(T)$ converging to some $\epsilon \in P(T)$ and note that the support of ϵ must be contained in the countable set $\bigcup_{n \neq m} (A(\lambda_n) \cap A(\lambda_m))$. $\square$

It is easy to see that, for T defined as above, the space $P(T)$ does not have ccc. Namely, setting $U_t = \{\epsilon \in P(T) : \epsilon(t) = 1\}$ for $t \in T$, we have that U_t is a nonempty open subset of $P(T)$ and that $U_t \cap U_s = \emptyset$, when $t, s \in T$ are incomparable.

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