

NONTRIVIAL TWISTED SUMS FOR FINITE HEIGHT SPACES UNDER MARTIN'S AXIOM

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ABSTRACT. We show that if we assume Martin's Axiom, then there exists a nontrivial twisted sum of c_0 and $C(K)$, for every compact space K with finite height and weight at least *continuum*. This result settles the problem of existence of nontrivial twisted sums of c_0 and $C(K)$, for finite height spaces K , under Martin's Axiom.

1. INTRODUCTION

In this paper we study the existence of nontrivial twisted sums of c_0 and $C(K)$, for compact and scattered spaces K . As usual, $C(K)$ denotes the space of continuous real-valued functions defined on a compact Hausdorff space K , endowed with the supremum norm. Recall that given Banach spaces X and Y a *twisted sum* of Y and X is an exact sequence in the category of Banach spaces of the form $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$, i.e., Z is a Banach space and the arrows are bounded operators. This twisted sum is said to be *trivial* if the image of $Y \rightarrow Z$ is complemented in Z . It is clear that there exist nontrivial twisted sums in the category of Banach spaces, since there are uncomplemented subspaces. An interesting problem is to determine if there are nontrivial twisted sums of Y and X , for a fixed pair of Banach spaces X and Y . In this context, the space c_0 plays a central role due to *Sobczyk's Theorem* [18] that states that c_0 is complemented in every separable superspace. Therefore, if X is a separable Banach space, then every twisted sum of c_0 and X is trivial. A natural question here is about the converse of last implication, i.e., if X is a Banach space such that every twisted sum of c_0 and X is trivial, then X is separable? This question is easily answered negatively, since there are nonseparable projective Banach spaces. However it becomes much more interesting in the context of $C(K)$ spaces. Recall that the Banach space $C(K)$ is separable if and only if the compact K is metrizable. Therefore, Sobczyk's Theorem ensures that if K is a metrizable compact space, then every twisted sum of c_0 and $C(K)$ is trivial. In this context the converse we are discussing can be rephrased

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as: Is there a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial? This question was initially proposed in [2] and it remains open. It has been recently studied in a series of papers [3, 6, 7, 8, 15]. In some of those papers, the class of compact spaces studied is the class of scattered spaces. Recall that a topological space $\mathcal{X}$ is called *scattered* if there exists an ordinal α such that its Cantor–Bendixson derivative $\mathcal{X}^{(\alpha)}$ is empty [13, Chapter 6]. The least ordinal α such that $\mathcal{X}^{(\alpha)} = \emptyset$ is called the *height* of $\mathcal{X}$ and we say that $\mathcal{X}$ has *finite height* if its height is a natural number. Surprisingly, additional set-theoretic assumptions have played an essential role in those works [5]. The main axioms that appear are the *Continuum Hypothesis* (CH) and *Martin’s Axiom* (MA). In [15], G. Plebanek and W. Marciszewski proved that, assuming $\text{MA} + \neg \text{CH}$, there exist nonmetrizable compact Hausdorff spaces K such that every twisted sum of c_0 and $C(K)$ is trivial; answering consistently the question we are discussing. More precisely, they showed that if we assume MA, then every twisted sum of c_0 and $C(K)$ is trivial, for every compact Hausdorff and separable space K with height 3 and $w(K) < \mathfrak{c}$, where $w(K)$ denotes the weight of K and $\mathfrak{c}$ denotes the cardinality of the *continuum* [15, Theorem 9.7]. In [6, Corollary 4.2], this result was generalized for any finite height. On the other hand, J. Castillo showed that if we assume CH, then there exists a nontrivial twisted sum of c_0 and $C(K)$, for every finite height and nonmetrizable compact space K [3, Theorem 1]. Therefore, the existence of nontrivial twisted sums of c_0 and $C(K)$, for finite height, separable and small compact spaces K is independent of the axioms of ZFC. This paper is a continuation of [6, 15]. Our main result (Theorem 3.5) states that if we assume $\text{MA} + \neg \text{CH}$, then there exists a nontrivial twisted sum of c_0 and $C(K)$, for every finite height space K with $w(K) \geq \mathfrak{c}$. A fundamental tool in the present work is the study of reminders of compactifications of the discrete space ω , where ω denotes the set of natural numbers. This study is presented in Section 2 and Section 3 is devoted to the construction of nontrivial twisted sums.

2. REMINDERS OF COMPACTIFICATIONS OF ω

Given a compactification $(\mathcal{Y}, \varphi)$ of a topological space $\mathcal{X}$, the *reminder* of this compactification is defined as $\mathcal{Y} \setminus \varphi[\mathcal{X}]$. Identifying $\varphi[\mathcal{X}]$ with $\mathcal{X}$, we denote the reminder by $\mathcal{Y} \setminus \mathcal{X}$. An interesting problem in the theory of compactifications and reminders is the problem of determining if a given compact Hausdorff space K is homeomorphic to the reminder of a compactification of the discrete space ω . The first deep result on this problem was obtained by I. Parovičenko [16] and says that every compact Hausdorff space K with $w(K) \leq \omega_1$ is homeomorphic to the reminder of a compactification of ω , where ω_1 denotes the first uncountable cardinal. It is well known that a compact space K is homeomorphic to the reminder of a compactification of ω if and only if K is a continuous image of $\beta\omega \setminus \omega$, where $\beta\omega$ denotes the Stone–Čech compactification of ω . Therefore, if we restrict ourselves

to the class of Boolean spaces, then *Stone's Duality* [13, Chapter 3] implies that a Boolean space K is homeomorphic to the remainder of a compactification of ω if and only if the algebra of clopen subsets of K embeds in $\wp(\omega)/\text{Fin}$. As usual $\wp(\omega)/\text{Fin}$ denotes the Boolean algebra obtained by quotienting the Boolean algebra $\wp(\omega)$ by its ideal Fin formed by the finite subsets of ω . Hence Parovičenko's Theorem implies that every Boolean algebra with cardinality at most ω_1 embeds in $\wp(\omega)/\text{Fin}$. Therefore, under CH, we have that every Boolean algebra with cardinality at most $\mathfrak{c}$ embeds in $\wp(\omega)/\text{Fin}$. This result does not hold in ZFC; for instance, it was shown in [12] that it is relatively consistent with ZFC that there exists a Boolean algebra with cardinality $\mathfrak{c}$ that does not embed in $\wp(\omega)/\text{Fin}$. Now, let us take a look at this problem under $\text{MA} + \neg\text{CH}$. In [9], it was shown that under MA every Boolean algebra with cardinality smaller than $\mathfrak{c}$ embeds in $\wp(\omega)/\text{Fin}$. However, the same does not hold for Boolean algebras with cardinality $\mathfrak{c}$. The Boolean algebra with cardinality $\mathfrak{c}$ that does not embed in $\wp(\omega)/\text{Fin}$ given in [12] is also relatively consistent with $\text{MA} + \neg\text{CH}$. To the best knowledge of the author, it is not known if it is relatively consistent with $\text{MA} + \neg\text{CH}$ that every Boolean algebra with cardinality $\mathfrak{c}$ embeds in $\wp(\omega)/\text{Fin}$. In this section, we present some contributions to the study of the problem of embedding Boolean algebras with cardinality $\mathfrak{c}$ in $\wp(\omega)/\text{Fin}$, under $\text{MA} + \neg\text{CH}$.

Let us start by fixing some notations. We denote by $|X|$ the cardinality of a set X . Given $A, B \in \wp(\omega)$ we say that A and B are *almost equal* if the symmetric difference $A \Delta B$ is finite and we denote this by $A =^* B$; we say that A and B are *almost disjoint* if $A \cap B =^* \emptyset$; we say that A is *almost contained* in B if $A \setminus B$ is finite and denote this by $A \subset^* B$. Given a collection $\mathcal{A}$ of subsets of ω , we denote by $\mathcal{I}(\mathcal{A})$ the ideal of $\wp(\omega)$ formed by the sets that are almost contained in a finite union of elements of $\mathcal{A}$.

Lemma 2.1. *Assume $\text{MA} + \neg\text{CH}$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset \wp(\omega)$ with $|\mathcal{A}| \leq \omega$, $|\mathcal{B}| < \mathfrak{c}$ and $|\mathcal{C}| < \mathfrak{c}$. Assume that:*

- (i) $A \cap B =^* \emptyset$, for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$;
- (ii) $\mathcal{C} \cap \mathcal{I}(\mathcal{A}) = \emptyset$;
- (iii) $\mathcal{C} \cap \mathcal{I}(\mathcal{B}) = \emptyset$.

Then there exists $S \in \wp(\omega)$ such that $A \subset^ S$, for every $A \in \mathcal{A}$, $B \cap S =^* \emptyset$, for every $B \in \mathcal{B}$, and $C \cap S$ and $C \cap (\omega \setminus S)$ are infinite, for every $C \in \mathcal{C}$.*

Proof. Let $\mathbb{P}$ denote the set of functions $f : \text{dom}(f) \subset \omega \rightarrow 2$ satisfying $f^{-1}(1) =^* \bigcup_{i=1}^n A_i$ and $f^{-1}(0) =^* \bigcup_{j=1}^m B_j$, where $n, m \in \omega$, $A_i \in \mathcal{A}$ and $B_j \in \mathcal{B}$. Endow $\mathbb{P}$ with the partial order defined as $f \leq g$ iff $g \subset f$. Note that if $f, g \in \mathbb{P}$ satisfy $f^{-1}(1) = g^{-1}(1)$, then they are compatible, therefore the fact that $\mathcal{A}$ is countable implies that $(\mathbb{P}, \leq)$ has ccc. Consider the following dense subsets of $\mathbb{P}$:

$$D_A = \{f \in \mathbb{P} : A \subset^* f^{-1}(1)\}, \quad D_B = \{f \in \mathbb{P} : B \subset^* f^{-1}(0)\} \text{ and} \\ D_{C,n} = \{f \in \mathbb{P} : \exists i, j > n \text{ with } i \in C \cap f^{-1}(1) \text{ and } j \in C \cap f^{-1}(0)\},$$

for every $A \in \mathcal{A}$, $B \in \mathcal{B}$, $C \in \mathcal{C}$ and $n \in \omega$. Let G be a filter in $\mathbb{P}$ that intersects all those dense sets whose existence is ensured by MA and define the function $\varphi = \bigcup G$. The desired set is given by $S = \varphi^{-1}(1)$. $\square$

Let $\mathfrak{B}$ be a Boolean algebra. We denote by $\langle X \rangle$ the subalgebra of $\mathfrak{B}$ generated by its subset X and by $\mathcal{I}(b)$ the *ideal generated* by $b \in \mathfrak{B}$, i.e., $\mathcal{I}(b) = \{c \in \mathfrak{B} : c \leq b\}$.

Theorem 2.2. *Assume MA + $\neg$ CH. Let $\mathfrak{B}$ be a Boolean algebra with $|\mathfrak{B}| = \mathfrak{c}$ and let X be a set of generators of $\mathfrak{B}$. If $|\{b \in X : |\mathcal{I}(b)| > \omega\}| < \mathfrak{c}$, then $\mathfrak{B}$ embeds in $\wp(\omega)/\text{Fin}$.*

Proof. If $\mathfrak{B}_0 = \langle \{b \in X : |\mathcal{I}(b)| > \omega\} \rangle$, then MA implies that there exists an embedding $\varphi : \mathfrak{B}_0 \rightarrow \wp(\omega)/\text{Fin}$ [9]. Enumerate $\{b \in X : |\mathcal{I}(b)| \leq \omega\}$ as $\{b_\alpha : \alpha \in \mathfrak{c}\}$. By recursion on $\alpha \in \mathfrak{c}$, we shall construct an increasing family $(\varphi_\alpha)_{\alpha \in \mathfrak{c}}$, where each φ_α is an embedding from $\langle \mathfrak{B}_0 \cup \{b_\beta : \beta \leq \alpha\} \rangle$ in $\wp(\omega)/\text{Fin}$ extending φ . Then the embedding of $\mathfrak{B}$ in $\wp(\omega)/\text{Fin}$ will be given by $\bigcup_{\alpha \in \mathfrak{c}} \varphi_\alpha$. For every $\alpha \in \mathfrak{c}$, set $\mathfrak{B}_\alpha = \langle \mathfrak{B}_0 \cup \{b_\beta : \beta < \alpha\} \rangle$. Assume that we have already defined φ_β , for every $\beta \in \alpha$, then $\psi = \bigcup_{\beta \in \alpha} \varphi_\beta$ is an embedding from $\mathfrak{B}_\alpha$ in $\wp(\omega)/\text{Fin}$ that extends φ . Recall that *Sikorski's extension Theorem* [13] ensures that there is an embedding of $\mathfrak{B}_{\alpha+1}$ into $\wp(\omega)/\text{Fin}$ extending ψ if and only if there is $S \in \wp(\omega)$ satisfying:

- (a) $\psi(b) \leq q(S)$ if and only if $b \leq b_\alpha$, for every $b \in \mathfrak{B}_\alpha$;
- (b) $\psi(b) \wedge q(S) = 0$ if and only if $b \wedge b_\alpha = 0$, for every $b \in \mathfrak{B}_\alpha$,

where $q : \wp(\omega) \rightarrow \wp(\omega)/\text{Fin}$ denotes the quotient map. Set:

$$\mathcal{A} = q^{-1}[\{\psi(b) : b \in \mathfrak{B}_\alpha \text{ and } b \leq b_\alpha\}],$$

$$\mathcal{B} = q^{-1}[\{\psi(b) : b \in \mathfrak{B}_\alpha \text{ and } b \wedge b_\alpha = 0\}] \text{ and}$$

$$\mathcal{C} = q^{-1}[\{\psi(b) : b \in \mathfrak{B}_\alpha, b \wedge b_\alpha \neq 0 \text{ and } b \wedge b'_\alpha \neq 0\}].$$

It is clear that $|\mathcal{B}| < \mathfrak{c}$, $|\mathcal{C}| < \mathfrak{c}$ and $|\mathcal{A}| \leq \omega$. Conditions (i), (ii) and (iii) of Lemma 2.1 follow from the fact that ψ is an embedding. Let S be the set given by Lemma 2.1 and note that S satisfies conditions (a) and (b) above. $\square$

In order to apply Theorem 2.2 to Boolean spaces, we present a simple characterization of sets of generators of the clopen algebra $\text{Clop}(K)$, for any Boolean space K . Recall that a collection $\mathfrak{F}$ of functions defined on a set X is said to *separate the points of X* if given $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists $f \in \mathfrak{F}$ such that $f(x_1) \neq f(x_2)$.

Lemma 2.3. *Let K be a Boolean space and $\{C_i : i \in I\} \subset \text{Clop}(K)$. Then $\{C_i : i \in I\}$ generates $\text{Clop}(K)$ if and only if $\{\chi_{C_i} : i \in I\}$ separates the points of K , where χ_{C_i} denotes the characteristic function of C_i .*

Proof. Let $\varphi : K \rightarrow 2^I$ be the continuous map whose i -th coordinate is χ_{C_i} and denote by $\varphi^* : \text{Clop}(2^I) \rightarrow \text{Clop}(K)$ the corresponding map under Stone's Duality. The conclusion is obtained by noting that the image of φ^*

is the subalgebra generated by $\{C_i : i \in I\}$ and that φ is injective if and only if $\{\chi_{C_i} : i \in I\}$ separates the points of K . $\square$

Given a topological space $\mathcal{X}$ and $x \in \mathcal{X}$, we define:

$$w(\mathcal{X}, x) = \min\{w(V) : V \text{ is a nhood of } x\}.$$

Proposition 2.4. *Assume $\text{MA} + \neg \text{CH}$. Let K be a Boolean space such that $w(K) = \mathfrak{c}$. If $w(\{p \in K : w(K, p) > \omega\}) < \mathfrak{c}$, then K is homeomorphic to the remainder of a compactification of ω .*

Proof. Define $F = \{p \in K : w(K, p) > \omega\}$ and note that F is a closed subset of K . For each $C \in \text{Clop}(F)$, choose an element $\tilde{C}$ in $\text{Clop}(K)$ such that $C = \tilde{C} \cap F$ and set $X = \{\tilde{C} : C \in \text{Clop}(F)\} \cup \{C \in \text{Clop}(K) : C \subset K \setminus F\}$. It is clear that $\mathcal{I}(C) = \text{Clop}(C)$ is countable, for every $C \in \text{Clop}(K)$ with $C \subset K \setminus F$. The result follows from Theorem 2.2 and Lemma 2.3, since $|\text{Clop}(F)| = w(F) < \mathfrak{c}$ and $\{\chi_C : C \in X\}$ separates the points of K . $\square$

Remark 2.5. Recall that if a Boolean space is scattered, then its weight coincides with its cardinality and any of its closed subspaces is again a scattered Boolean space [13].

Corollary 2.6. *Assume $\text{MA} + \neg \text{CH}$. Let K be a compact Hausdorff and scattered space with $w(K) = \mathfrak{c}$. If $|\{p \in K : w(K, p) > \omega\}| < \mathfrak{c}$, then K is homeomorphic to the remainder of a compactification of ω .*

Proof. It follows from Remark 2.5 and Proposition 2.4. $\square$

Remark 2.7. Note that it does not hold in ZFC that every scattered compact space with weight at most $\mathfrak{c}$ is homeomorphic to the remainder of a compactification of ω . Indeed, K. Kunen [14] showed that it is relatively consistent with ZFC that $\omega_2 \leq \mathfrak{c}$ and that the ordinal segment $[0, \omega_2]$ is not homeomorphic to the remainder of a compactification of ω . However, Kunen's result is not relatively consistent with $\text{MA} + \neg \text{CH}$. It seems to be unknown if, under $\text{MA} + \neg \text{CH}$, every scattered compact space with weight $\mathfrak{c}$ is homeomorphic to the remainder of a compactification of ω . Therefore, Corollary 2.6 is an interesting contribution to this open problem.

3. NONTRIVIAL TWISTED SUMS

Let us now explain the relationship between reminders of compactifications of ω and twisted sums of Banach spaces. If $\gamma\omega$ is a compactification of ω , then we have a natural isometric copy of c_0 in $C(\gamma\omega)$; this copy consists of the elements of $C(\gamma\omega)$ that vanish in the reminder $\gamma\omega \setminus \omega$. Following [15], we say that a compactification $\gamma\omega$ of ω is *tame* if this natural copy of c_0 is complemented in $C(\gamma\omega)$. Therefore, if the compactification $\gamma\omega$ is not tame, then there exists a nontrivial twisted sum of c_0 and $C(\gamma\omega \setminus \omega)$, since $C(\gamma\omega)$ quotiented by the natural copy of c_0 is isometric to $C(\gamma\omega \setminus \omega)$. In [15, Theorem 2.8], the following necessary condition for the tameness of a compactification is presented.

Lemma 3.1. *If a compactification $\gamma\omega$ of ω is tame, then its reminder $\gamma\omega \setminus \omega$ carries a strictly positive measure.*

It is well known that if K is a nonseparable scattered compact space, then K does not carry a strictly positive measure. Using this and Parovičenko's Theorem, it was shown in [15, Corrolary 9.6] that under CH there exists a nontrivial twisted sum of c_0 and $C(K)$, for every nonseparable compact scattered space K with $w(K) = \mathfrak{c}$. In the next proposition, we present a version of this result under $\text{MA} + \neg \text{CH}$.

Proposition 3.2. *Assume $\text{MA} + \neg \text{CH}$. Let K be a compact Hausdorff and scattered space with $w(K) < \mathfrak{c}$. If K is nonseparable, then there exists a nontrivial twisted sum of c_0 and $C(K)$.*

Proof. It follows from the discussion above, Lemma 3.1 and the fact that $\text{MA} + \neg \text{CH}$ implies that every compact space K with $w(K) < \mathfrak{c}$ is homeomorphic to the reminder of a compactification of ω [9]. $\square$

Another class of compact spaces discussed in [15] that does not carry a strictly positive measure is the class of nonseparable compact lines. Recall that a *compact line* is a totally ordered space that is compact, when endowed with the order topology. It was shown in [15, Theorem 8.1] that if we assume CH, then there exists a nontrivial twisted sum of c_0 and $C(K)$, for every nonseparable compact line K . In Proposition 3.3, we prove that this actually holds in ZFC.

Proposition 3.3. *If K is a nonseparable compact line, then there exists a nontrivial twisted sum of c_0 and $C(K)$.*

Proof. The case when K does not have ccc was already solved in [15, Theorem 8.1]. If K has ccc, then K is first-countable [10, 3.12.4]. In [1, Corollary 3.2], M. Bell proved that every first-countable compact line is homeomorphic to the reminder of a compactification of ω , therefore the result follows from the fact that K does not carry a strictly positive measure [17] and from Lemma 3.1. $\square$

A fundamental ingredient to establish our main result (Theorem 3.5) is Lemma 3.4 below. This result is an adaptation of [15, Theorem 9.1]. It is interesting to observe that even though the proof of [15, Theorem 9.1] is purely topological, we obtained Lemma 3.4 using homological tools, inspired by [3]. For the definitions of the homological objects that appear in the proof of Lemma 3.4, see [3, 4]. Given topological spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $C(\mathcal{X}, \mathcal{Y})$ the set of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$. As usual, $\text{dens}(\mathcal{X})$ denotes the density of $\mathcal{X}$. Moreover, given Banach spaces X and Y , we denote by $\mathcal{L}(X, Y)$ the space of bounded operators from X to Y .

Lemma 3.4. *Let K be a compact Hausdorff and scattered space. Assume that there exists a closed subset F of K satisfying:*

- (a) *F is homeomorphic to the reminder of a compactification of ω ;*

(b) $|C(F, F)| > \mathfrak{c}$.

Then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Proof. Note that condition (a) implies that $w(F) \leq \mathfrak{c}$, hence Remark 2.5 ensures that $|F| \leq \mathfrak{c}$. Initially, we will show that $|\text{Ext}(C(F), c_0)| > \mathfrak{c}$. It holds that $|\text{Ext}(C(F), c_0)| = |\mathcal{L}(C(F), \ell_\infty/c_0)/W|$, where W denotes the closed subspace of $\mathcal{L}(C(F), \ell_\infty/c_0)$ consisting of the operators that admit a lifting to ℓ_∞ [4, Proposition 1.4.f]. To establish our estimate, we will prove that $|\mathcal{L}(C(F), \ell_\infty/c_0)| > \mathfrak{c}$ and that $|W| \leq \mathfrak{c}$. Clearly $|W| \leq |\mathcal{L}(C(F), \ell_\infty)|$ and it is easy to see that $|\mathcal{L}(C(F), \ell_\infty)| \leq |C(F)^*|^\omega$. Therefore we conclude that $|W| \leq \mathfrak{c}$, since $|C(F)^*| \leq \mathfrak{c}$ [11, Theorem 12.28]. Let $\varphi : \beta\omega \setminus \omega \rightarrow F$ be the continuous onto map given by (a) and note that the following maps are injective:

$$C(F, F) \xrightarrow{T} C(\beta\omega \setminus \omega, F) \xrightarrow{S} \mathcal{L}(C(F), C(\beta\omega \setminus \omega)),$$

where $T(f) = f \circ \varphi$, $S(\phi) = \phi^*$ and $\phi^* : C(F) \rightarrow C(\beta\omega \setminus \omega)$ is given by $\phi^*(g) = g \circ \phi$. Then condition (b) implies that $|\mathcal{L}(C(F), \ell_\infty/c_0)| > \mathfrak{c}$. Now consider the restriction operator $R : C(K) \rightarrow C(F)$. It follows from [3, Lemma 4] that there exist a closed subspace X of $C(K)$ and a closed subspace Y of $\text{Ker}(R)$ such that $\text{dens}(Y) \leq \text{dens}(C(F))$ and such that the diagram below has exact rows and commutes:

$$(1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(R) & \hookrightarrow & C(K) & \xrightarrow{R} & C(F) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \text{identity} \\ 0 & \longrightarrow & Y & \hookrightarrow & X & \xrightarrow{R|_X} & C(F) \longrightarrow 0 \end{array}$$

Assuming by contradiction that $\text{Ext}(C(K), c_0)$ is zero, one can argue as in the proof of [3, Lemma 5] using diagram (1) to obtain a surjective map from $\mathcal{L}(Y, c_0)$ to $\text{Ext}(C(F), c_0)$. However this is a contradiction, because:

$$|\mathcal{L}(Y, c_0)| \leq |Y^*|^\omega \leq \mathfrak{c},$$

where the last inequality follows from the fact that $\text{dens}(Y^*) = \text{dens}(Y)$, since $C(K)$ is Asplund (see [11, Theorem 12.29] and [19, Theorem 6]). $\square$

We are now ready to prove our main result.

Theorem 3.5. *Assume $\text{MA} + \neg \text{CH}$. If K is a compact Hausdorff and scattered space with finite height and $w(K) \geq \mathfrak{c}$, then there exists a nontrivial twisted sum of c_0 and $C(K)$.*

Proof. Denote by N the height of K . Firstly, let us prove that we can assume that $K^{(N-1)}$ has only one point. Write $K^{(N-1)} = \{p_1, \dots, p_k\}$, where $k = |K^{(N-1)}|$. Clearly, there exist disjoint clopen subsets $C_1, \dots, C_k$ of K such that $K = \bigcup_{i=1}^k C_i$ and $p_i \in C_i$, for every $i = 1, \dots, k$. It is easy to see that $C_i^{(N-1)} = \{p_i\}$, for every $i = 1, \dots, k$ and that there exists an

index i_0 such that $w(C_{i_0}) \geq \mathfrak{c}$. Since the canonical copy of $C(C_{i_0})$ in $C(K)$ is complemented, a nontrivial twisted sum of c_0 and $C(C_{i_0})$ yields a nontrivial twisted sum of c_0 and $C(K)$. Set $I_j = K^{(j)} \setminus K^{(j+1)}$, for $j = 0, \dots, N-2$. For each j and each $\gamma \in I_j$, since γ is isolated in $K^{(j)}$, we can choose $S_\gamma^j \in \text{Clop}(K)$ with $S_\gamma^j \cap K^{(j)} = \{\gamma\}$ and $w(S_\gamma^j) = |S_\gamma^j| = w(K, \gamma)$. We proceed by induction on the height of K . Since $w(K) \geq \mathfrak{c}$, the first step of the inductive process is $N = 2$. In this case K is the one-point compactification of a discrete space I of size greater or equal to $\mathfrak{c}$. It is well known that $C(K)$ is isomorphic to $c_0(I)$ and that there exists a nontrivial twisted sum of c_0 and $c_0(I)$ [3, §2]. Now fix $N \geq 3$ and note that if there exist $j \leq N-2$ and $\gamma \in I_j$ such that $|S_\gamma^j| \geq \mathfrak{c}$, then the induction hypothesis implies that there exists a nontrivial twisted sum of c_0 and $C(S_\gamma^j)$. Since $C(S_\gamma^j)$ is complemented in $C(K)$, it follows that there exists a nontrivial twisted sum of c_0 and $C(K)$. Now assume that $|S_\gamma^j| < \mathfrak{c}$, for every $j \leq N-2$ and every $\gamma \in I_j$. Let j be the greatest natural number with $|I_j| \geq \mathfrak{c}$. and fix a subset I of I_j with $|I| = \mathfrak{c}$. Consider the closed subset $F = K^{(j+1)} \cup I$ of K and note that

$$|\{p \in F : w(F, p) > \omega\}| < \mathfrak{c}.$$

Therefore, Corollary 2.6 ensures that F is homeomorphic to the remainder of a compactification of ω . Finally, to conclude the result using Lemma 3.4, let us prove that $|C(F, F)| > \mathfrak{c}$. Set $A = I \setminus \bigcup_{k=j+1}^{N-2} \bigcup_{\gamma \in I_k} S_\gamma^k$ and note that $|A| = \mathfrak{c}$, since $\mathfrak{c}$ is regular under MA. Moreover any bijection $h : A \rightarrow A$ induces a continuous map $\varphi_h : F \rightarrow F$ that fixes every point of $F \setminus A$ and such that $\varphi_h|_A = h$. $\square$

Remark 3.6. Note that the problem of existence of nontrivial twisted sums of c_0 and $C(K)$, for finite height spaces K is now settled under MA. Indeed, in [3], it was solved under CH. Under $\text{MA} + \neg \text{CH}$, if $w(K) < \mathfrak{c}$, then the separable case was solved in [6, 15] and the nonseparable case is solved by Proposition 3.2. Finally, the case when $w(K) \geq \mathfrak{c}$ is solved by Theorem 3.5.

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