EXTENSION PROPERTY AND COMPLEMENTATION OF ISOMETRIC COPIES OF CONTINUOUS FUNCTIONS SPACES

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Abstract. We prove that every isometric copy of $C(L)$ in $C(K)$ is complemented if L is a compact Hausdorff space of finite height and K is a compact Hausdorff space satisfying the *extension property*, i.e., every closed subset of K admits an extension operator. The space $C(L)$ can be replaced by its subspace $C(L|F)$ consisting of functions that vanish on a closed subset F of L. We also study the class of spaces having the extension property, establishing some stability results for this class and relating it to other classes of compact spaces.

1. Introduction

In this paper we prove a result (Theorem 2.5) concerning the complementation of isometric copies of $C(L)$ in $C(K)$, for compact Hausdorff spaces K and L satisfying certain assumptions. Our result concerns, in particular, the complementation of Banach subalgebras of $C(K)$ (Corollary 2.6). Indeed, recall that a Banach subalgebra (with unity) of $C(K)$ is the range $\phi^*[C(L)]$ of the linear isometric embedding $\phi^* : C(L) \ni f \longmapsto f \circ \phi =: \phi^*(f) \in C(K)$, where $\phi : K \to L$ is a continuous surjection. The Banach subalgebra $\phi^*[C(L)]$ is complemented in $C(K)$ if and only if ϕ admits an *averaging operator*, i.e., a bounded linear left inverse $A : C(K) \to C(L)$ for the composition operator ϕ^*. Finding conditions under which a continuous surjection admits an averaging operator is a classical problem in Banach space theory ([1, 5, 13]).

More generally, Theorem 2.5 deals with the complementation of isometric copies in $C(K)$ of the subspace $C(L|F)$ of $C(L)$ consisting of continuous functions vanishing on a closed subset F of L.

We observe that the special case of Corollary 2.6 when K and L are compact lines and $\phi : K \to L$ is a continuous *increasing* surjection is a corollary of [8, Lemma 2.7], though the general case (when ϕ is not necessarily increasing) is new. By a *compact line* we mean a linearly ordered set which is compact in the order topology.

Date: January 24th, 2014.

2010 *Mathematics Subject Classification.* 46B20,46E15,54G12.

Key words and phrases. Banach spaces of continuous functions; extension and averaging operators; complementation of subspaces.

The first author is sponsored by FAPESP (Process no. 2012/25171-0).

The assumptions imposed on the spaces K and L in Theorem 2.5 are the following: the space L is assumed to be scattered, of finite (Cantor–Bendixson) height, and the space K is required to satisfy a condition which we call the extension property. We say that K has the *extension property* if every closed subset F of K admits an extension operator in K. By an *extension operator* for F in K we mean a bounded linear map $E_F : C(F) \to C(K)$ such that $E_F(f)$ is an extension of f to K, for all $f \in C(F)$.

We note that, if K and L satisfy the assumptions of Theorem 2.5 and $C(K)$ contains an isometric copy of $C(L)$, then the space $C(L)$ is actually isomorphic to $c_0(I)$, for some index set I (Corollary 3.11). Thus, our Theorem 2.5 can be seen as a Sobczyk-like theorem. The complementation of isomorphic copies of $c_0(I)$ in Banach spaces has been studied in [2]. More precisely, in [2, Theorem 1.1] it is shown that every isomorphic copy of $c_0(I)$ in a Banach space X is complemented if $|I| < \aleph_\omega$ and if X is isomorphic to a subspace of a $C(K)$ space, with K a Valdivia compactum. We observe that our Theorem 2.5 does not follow from [2, Theorem 1.1], even when $C(L) \cong c_0(I)$ with $|I| < \aleph_\omega$. Namely, if K is a separable nonmetrizable compact line (say, the double arrow space), then K has the extension property, yet $X = C(K)$ does not satisfy the assumptions of [2, Theorem 1.1]. The latter statement follows from [4, Theorem 2.1] and from the fact that the space of continuous functions on a Valdivia compactum has the separable complementation property ([14, Lemma, pg. 494]).

This paper is organized as follows. Section 2 is devoted to the proof of Theorem 2.5. In Section 3 we study the extension property. We start the section by stating results that are immediate consequences of well-known facts. Then we investigate some stability properties of the class of spaces having the extension property. We conclude the section with examples of concrete spaces and of classes of spaces that do not have the extension property.

2. Complementation of isometric copies of $C(L)$ in $C(K)$

In the paper, all topological spaces are assumed to be Hausdorff. By an *operator* we always mean a bounded linear map between Banach spaces. A subspace of a Banach space X is said to be *complemented in* X if it is equal to the range of a bounded linear projection $P : X \to X$. The symbol $C(K)$ denotes the Banach space of real-valued continuous functions on a compact space K, endowed with the supremum norm; by $\mathbf{1}_K$ we denote the function on K which is constant and equal to 1. If F is a closed subset of K, we denote by $\rho_F : C(K) \ni f \longmapsto f|_F =: \rho_F(f) \in C(F)$ the restriction operator and by $C(K|F)$ its kernel.

Let us first prove a simple fact that will be used in many places throughout the paper.

Lemma 2.1. *Let X and Y be Banach spaces and $T : X \to Y$ be an operator. Then:*

(a) *T admits a bounded linear left inverse if and only if T is injective and the range of T is complemented in Y;*
(b) *T admits a bounded linear right inverse if and only if T is surjective and the kernel of T is complemented in X.*

Proof. If T admits a bounded linear left inverse $S : Y \to X$, then T is obviously injective and $T \circ S$ is a bounded projection onto the range of T. Conversely, if T is injective and $P : Y \to T[X]$ is a bounded projection, then $S = T^{-1} \circ P$ is a bounded linear left inverse for T.

If T admits a bounded linear right inverse $S : Y \to X$, then T is obviously surjective and $\mathrm{Id}_X - S \circ T$ is a bounded projection onto the kernel of T. Conversely, if T is surjective and $X = \mathrm{Ker}(T) \oplus Z$, for some closed subspace Z of X, then $(T|_Z)^{-1}$ is a bounded linear right inverse for T. □

Corollary 2.2. *Let K be a compact space and F be a closed subset of K. Then F admits an extension operator in K if and only if $C(K|F)$ is complemented in $C(K)$.*

Proof. Simply observe that an extension operator for F in K is the same as a bounded linear right inverse for ρ_F. □

For the proof of the main result of this section, we need two lemmas.

Lemma 2.3. *Let X and Y be Banach spaces, $T : X \to Y$ be an operator, and Z be a closed subspace of X. If the space $T[Z]$ is complemented in Y and $Z \cap \mathrm{Ker}(T)$ is complemented in X, then Z is complemented in X.*

Proof. Let P be a projection of X onto $Z \cap \mathrm{Ker}(T)$ and Q be a projection of Y onto $T[Z]$. Since the operator $(\mathrm{Id}_X - P)|_Z : Z \to Z$ annihilates the kernel of $T|_Z$ and $T[Z]$ is closed, there exists a unique operator $M : T[Z] \to Z$ such that $M \circ T|_Z = (\mathrm{Id}_X - P)|_Z$. It is easily checked that $M \circ Q \circ T + P$ is a projection of X onto Z. □

Lemma 2.4. *Let K be a compact space, I be a set and $S : c_0(I) \to C(K)$ be a linear isometric embedding. Let $\{\xi_i : i \in I\} \subset C(K)$ be the image under S of the canonical basis of $c_0(I)$ and set $H = \bigcap_{i \in I} \xi_i^{-1}(0)$. Then the range of S is complemented in $C(K|H)$.*

Proof. Since S is an isometric embedding, we have $\|\xi_i\| = 1$ and thus, for each $i \in I$, there exists $x_i \in K$ with $|\xi_i(x_i)| = 1$. The fact that S is a linear isometric embedding yields that $\|\xi_i \pm \xi_j\| = 1$ for $i \neq j$ and therefore $\xi_j(x_i) = 0$ for $i \neq j$. It follows that the points x_i are distinct and that all the points of accumulation of the set $\{x_i : i \in I\}$ are in H. This implies that, for $\xi \in C(K|H)$, the family $\big(\xi(x_i)\big)_{i \in I}$ is in $c_0(I)$. Hence the map $U : C(K|H) \to c_0(I)$ defined by $U(\xi) = \big(\xi_i(x_i)\xi(x_i)\big)_{i \in I}$ is a bounded linear left inverse for S and thus, by Lemma 2.1, the range of S is complemented in $C(K|H)$. □

The *Cantor–Bendixson derivative* of a topological space K is the set K' of accumulation points of K. The n-th Cantor–Bendixson derivative of K, denoted $K^{(n)}$, is defined recursively by $K^{(0)} = K$, $K^{(n+1)} = (K^{(n)})'$. The space K is said to be of *finite height* if $K^{(n)} = \emptyset$, for some natural number n; in this case, the *height* of K is the least such number.

Theorem 2.5. *Let K be a compact space having the extension property, L be a compact space of finite height and F be a closed subset of L. Then every isometric copy of $C(L|F)$ in $C(K)$ is complemented.*

Proof. We will use induction on the height of L. If L has height zero, then L is empty and the thesis holds trivially. Now assume that L has positive height and that the theorem holds for spaces of height smaller than the height of L.

Let $S : C(L|F) \to C(K)$ be a linear isometric embedding and define $R : C(L|F) \to C(L'|L'\cap F)$ by setting $R(f) = f|_{L'}$, for all $f \in C(L|F)$. Note that R is surjective and that $\mathrm{Ker}(R) = C(L|L'\cup F)$. Setting $I = L\setminus(L'\cup F)$, then $f \mapsto f|_I$ is an isometry from $C(L|L' \cup F)$ onto $c_0(I)$. For $i \in I$, denote by $f_i \in C(L|L' \cup F)$ the characteristic function of $\{i\}$ and set $\xi_i = S(f_i)$.

Define H as in the statement of Lemma 2.4. The map $\rho_H \circ S$ annihilates the set $\big\{f_i : i \in I\big\}$ and thus also its closed span $C(L|L' \cup F) = \mathrm{Ker}(R)$. Since $\rho_H \circ S$ annihilates $\mathrm{Ker}(R)$ and R is surjective, there exists a unique operator $\widetilde{S} : C(L'|L' \cap F) \to C(H)$ such that $\widetilde{S} \circ R = \rho_H \circ S$.

We will prove in a moment that $\widetilde{S}$ is an isometric embedding. Assuming that this is the case, we will conclude the proof by applying Lemma 2.3 with $T = \rho_H$ and with Z equals to the range of S. Note that $T[Z]$ equals the range of $\widetilde{S}$, which is complemented in $C(H)$, by the induction hypothesis. Moreover, $Z \cap \mathrm{Ker}(T) = S[C(L|L' \cup F)]$ is complemented in $C(K|H)$ by Lemma 2.4 and $C(K|H)$ is complemented in $C(K)$, because K has the extension property (Corollary 2.2).

Now let us prove that $\widetilde{S}$ is an isometric embedding. Let $g \in C(L'|L' \cap F)$ with $\|g\| = 1$ be fixed. Then $\widetilde{S}(g) = S(f)|_H$, where $f \in C(L|F)$ is an arbitrarily chosen extension of g. To prove that $\|\widetilde{S}(g)\| = 1$, we will show that f can be chosen such that $\|f\| = 1$ and such that $S(f)$ attains its maximum modulus on H. Consider first the case when g admits an extension $f \in C(L|F)$ with $|f| < 1$ on I. Assume by contradiction that $S(f)$ attains its maximum modulus at a point $x \in K \setminus H$. Then $\xi_i(x) \neq 0$, for some $i \in I$. For $\varepsilon > 0$ small enough, we have $\|f \pm \varepsilon f_i\| = 1$ and hence

$$\begin{aligned} 1 = \max\{&\|S(f + \varepsilon f_i)\|, \|S(f - \varepsilon f_i)\|\} \\ &\geq \max\{|S(f)(x) + \varepsilon\xi_i(x)|, |S(f)(x) - \varepsilon\xi_i(x)|\} > 1. \end{aligned}$$

Finally, let $f \in C(L|F)$ be an extension of g with $\|f\| = 1$. We can assume that the set

$$J = \big\{i \in I : |f(i)| = 1\big\}$$

is infinite, otherwise f could be modified so that $|f| < 1$ on I. For $i \in J$, we have $\max\{\|f + f_i\|, \|f - f_i\|\} = 2$ and thus there exists $x_i \in K$ with $\max\{|S(f)(x_i) + \xi_i(x_i)|, |S(f)(x_i) - \xi_i(x_i)|\} = 2$. Since $|S(f)(x_i)| \leq 1$ and $|\xi_i(x_i)| \leq 1$, we must have $|S(f)(x_i)| = 1$ and $|\xi_i(x_i)| = 1$. Note that $\|\xi_i \pm \xi_j\| = 1$ for distinct $i, j \in J$. Hence, as in the proof of Lemma 2.4, the points x_i, $i \in J$, are distinct and the points of accumulation of $\big\{x_i : i \in J\big\}$ are in H. Then, picking an accumulation point, say x, of the (infinite) set $\big\{x_i : i \in J\big\}$, we obtain $x \in H$ and $|S(f)(x)| = 1$, concluding the proof. □

Corollary 2.6. *Let K and L be compact spaces and $\phi : K \to L$ be a continuous surjective map. If K has the extension property and L is of finite height, then $\phi^*[C(L)]$ is complemented in $C(K)$ (equivalently, ϕ admits an averaging operator).* □

The following result is an immediate consequence of Theorem 2.5. It is obtained by letting $L = I \cup \{\infty\}$ be the one-point compactification of a discrete space I and by setting $F = \{\infty\}$. Such a result has been stated in [12, Theorem 2.1] in the case when I is countable and K is the double arrow space, though the proof that appears in [12] works also in the general case.

Corollary 2.7. *If K is a compact space having the extension property, then every isometric copy of $c_0(I)$ in $C(K)$ is complemented.* □

Remark 2.8. Consider the class $\mathfrak{C}$ of compact spaces K such that, for every compact space L of finite height and every closed subset F of L, it holds that every isometric copy of $C(L|F)$ in $C(K)$ is complemented. Theorem 2.5 states that this class contains all compact spaces having the extension property. The class $\mathfrak{C}$ is clearly closed under continuous images; namely, this follows from the observation that if a compact space K is a continuous image of a compact space $\widetilde{K}$, then there exists a linear isometric embedding of $C(K)$ in $C(\widetilde{K})$.

3. The extension property

In this section we study the class of compact spaces having the (regular) extension property. Given a closed subset F of a compact space K, we call an extension operator $E_F : C(F) \to C(K)$ *regular* if $\|E_F\| = 1$ and $E_F(\mathbf{1}_F) = \mathbf{1}_K$. By [13, Proposition 1.2], E_F is regular if and only if $E_F(\mathbf{1}_F) = \mathbf{1}_K$ and E_F is a positive operator. If every nonempty closed subset of K admits a regular extension operator in K, we say that K has the *regular extension property.* Clearly the (regular) extension property is hereditary to closed subspaces. We start by collecting some well-known results about the existence of extension operators.

In [13, §6], a compact space K is called a *Dugundji space* if for every compact space L containing (a homeomorphic copy of) K, there exists a regular extension operator for K in L. It is well-known that every nonempty compact metrizable space is a Dugundji space (see the proof of [13, Theorem 6.6]). The following proposition is an immediate consequence of this fact.

Proposition 3.1. *Every compact metrizable space has the regular extension property.* □

In a compact line, a regular extension operator for a nonempty closed subset can be constructed by decomposing its complement into disjoint open intervals and by using Urysohn's Lemma. One thus obtain the following result.

Proposition 3.2. *Every compact line has the regular extension property.*

Proof. See [9, Lemma 4.2]. □

A standard argument using partitions of unity shows that the (regular) extension property is local.

Proposition 3.3. *Let K be a compact space and F be a nonempty closed subset of K. If every point $x \in F$ has a closed neighborhood V in K such that $V \cap F$ admits an extension operator (resp., a regular extension operator) in V, then F admits an extension operator (resp., a regular extension operator) in K.*

Proof. See [13, Lemma 3.6]. □

Corollary 3.4. *Let K be a compact space. If every point of K has a closed neighborhood having the (resp., regular) extension property, then K has the (resp., regular) extension property.* □

3.1. **Stability properties of the class of spaces having the extension property.** We will first establish that, under some additional assumptions, a continuous image of a space having the extension property also has the extension property (Propositions 3.5 and 3.10). We leave open the question of whether the class of spaces having the extension property is closed under continuous images. Next, we show that the class of spaces having the regular extension property is closed under the operation of taking Alexandroff compactifications of the topological sums of arbitrarily many spaces (Proposition 3.12).

Proposition 3.5. *Let $\phi : K \to L$ be a continuous surjection, where K and L are compact spaces. Assume that ϕ admits a regular averaging operator $A : C(K) \to C(L)$ (i.e., A is an averaging operator with $\|A\| = 1$). If K has the (resp., regular) extension property, then L also has the (resp., regular) extension property.*

Proof. Given a nonempty closed subset F of L, let $H = \phi^{-1}[F]$ and E_H be a (regular) extension operator for H in K. Set

$$E_F = A \circ E_H \circ (\phi|_H)^* : C(F) \longrightarrow C(L).$$

That E_F is indeed an extension operator follows easily from the fact that, for all $x \in L$, there exists a probability measure μ_x on $\phi^{-1}(x)$ such that $A(h)(x) = \int_{\phi^{-1}(x)} h \, \mathrm{d}\mu_x$, for all $h \in C(K)$ (see [13, Proposition 4.1]). $\square$

The following definition and lemma are required for the proof of Propositions 3.8 and 3.10.

Definition 3.6. Given a compact space K, a closed subset F of K, and a closed subset H of F, by an *extension operator modulo* H for F in K we mean an operator which is a right inverse for the map $\rho_F|_{C(K|H)} : C(K|H) \to C(F|H)$.

By Lemma 2.1, F admits an extension operator modulo H in K if and only if $C(K|F)$ is complemented in $C(K|H)$.

Lemma 3.7. *Let K be a compact space, F be a closed subset of K, and H be a closed subset of F. If F admits an extension operator modulo H in K and H admits an extension operator in K, then F admits an extension operator in K.*

Proof. Since F admits an extension operator modulo H in K, the subspace $C(K|F)$ is complemented in $C(K|H)$. Moreover, since H admits an extension operator in K, the subspace $C(K|H)$ is complemented in $C(K)$. Hence $C(K|F)$ is complemented in $C(K)$. $\square$

Proposition 3.8. *Let K be a compact space. If its Cantor–Bendixson derivative K' has the extension property and K' admits an extension operator in K, then K has the extension property.*

Proof. Assume that K' has the properties appearing in the statement. Let F be a closed subset of K. Then $F \cap K'$ admits an extension operator in K. Moreover, the map $E : C(F|F \cap K') \to C(K|F \cap K')$ defined by $E(f)|_F = f$, $E(f)|_{K \setminus F} \equiv 0$, for $f \in C(F|F \cap K')$, is an extension operator modulo $F \cap K'$ for F in K. The conclusion now follows from Lemma 3.7 by putting $H = F \cap K'$. $\square$

Corollary 3.9. *Let K be a compact space of finite height and assume that, for all n, the $(n+1)$-th Cantor–Bendixson derivative $K^{(n+1)}$ admits an extension operator in $K^{(n)}$. Then K has the extension property.* $\square$

Proposition 3.10. *Let K be a compact space of finite height. If K is a continuous image of a compact space having the extension property, then K has the extension property and the space $C(K)$ is isomorphic to $c_0(K)$.*

Proof. Note that $C(K|K')$ is isometric to $c_0(K \setminus K')$ and therefore it is complemented in $C(K)$, by Remark 2.8. Thus $C(K) \cong c_0(K \setminus K') \oplus C(K')$ and, by Corollary 2.2, K' admits an extension operator in K. The conclusion is now obtained using induction on the height of K and Proposition 3.8. $\square$

Corollary 3.11. *If K is a compact space having the extension property and L is a compact space of finite height such that $C(K)$ contains an isometric copy of $C(L)$, then L has the extension property and hence the space $C(L)$ is isomorphic to $c_0(L)$.*

Proof. Observe that, by Holsztyński's Theorem [7], if $C(K)$ contains an isometric copy of $C(L)$, then L is a continuous image of a closed subspace of K. In particular, L is a continuous image of a compact space having the extension property, and hence the conclusion follows from Proposition 3.10. □

By the *Alexandroff compactification* of a locally compact space X we mean X itself, if X is compact, or its one-point compactification $X \cup \{\infty\}$, if X is not compact.

Proposition 3.12. *The class of compact spaces having the regular extension property is closed under the operation of taking the Alexandroff compactification of the topological sums of arbitrarily many spaces.*

Proof. Let $(K_i)_{i\in I}$ be a family of nonempty compact spaces having the regular extension property. Denote by $\dot\bigcup_{i\in I} K_i$ its topological sum and by K the Alexandroff compactification of the latter. Let F be a nonempty closed subset of K and set $F_i = F \cap K_i$, for every $i \in I$. When $F_i \neq \emptyset$, let E_i be a regular extension operator for F_i in K_i. We shall define a regular extension operator E_F for F in K as follows.

If F is contained in $\dot\bigcup_{i\in I} K_i$, then F is contained in a union of finitely many K_i's. In this case, choose some $p \in F$ and for $f \in C(F)$, define $E_F(f)$ by setting $E_F(f)|_{K_i} = E_i(f|_{F_i})$, if $F_i \neq \emptyset$, $E_F(f)|_{K_i} \equiv f(p)$, if $F_i = \emptyset$, and (if I is infinite) $E_F(f)(\infty) = f(p)$. If ∞ belongs to F, define $E_F(f)$ by setting $E_F(f)|_{K_i} = E_i(f|_{F_i})$, if $F_i \neq \emptyset$, $E_F(f)|_{K_i} = f(\infty)$, if $F_i = \emptyset$, and $E_F(f)(\infty) = f(\infty)$. In the case when $\infty \in F$, the continuity of $E_F(f)$ at the point ∞ is easily proven using the fact that E_i is a positive operator. □

Corollary 3.13. *If K is a compact space of height 2, then K has the regular extension property.*

Proof. Such a K is the topological sum of finitely many one-point compactifications of discrete spaces. □

3.2. **Negative results about the extension property.** We now give a few examples of spaces for which the extension property fails.

Example 3.14. The *ladder system space* [3, pg. 164] is an example of a compact space of height 3 that does not have the extension property. We recall relevant definitions. Denote by ω_1 the first uncountable ordinal and by $L(\omega_1)$ the subset of ω_1 consisting of limit ordinals. Set $S(\omega_1) = \omega_1 \backslash L(\omega_1)$. A *ladder system* on ω_1 is a family $\mathcal{S} = (s_\alpha)_{\alpha\in L(\omega_1)}$ where, for each $\alpha \in L(\omega_1)$, we have $s_\alpha = \{s_\alpha^n : n \in \omega\}$ and $(s_\alpha^n)_{n\in\omega}$ is a strictly increasing sequence in $S(\omega_1)$ order-converging to α. The *ladder system topology* $\tau_{\mathcal{S}}$ on ω_1 is the one for which the elements of $S(\omega_1)$ are isolated and the fundamental neighborhoods of $\alpha \in L(\omega_1)$ are unions of $\{\alpha\}$ with sets cofinite in s_α. The *ladder system space* is the one-point compactification $K_{\mathcal{S}} = \omega_1^{\mathcal{S}} \cup \{\infty\}$, where $\omega_1^{\mathcal{S}} = (\omega_1, \tau_{\mathcal{S}})$. We claim that the closed subset $F = L(\omega_1) \cup \{\infty\}$ does not admit an extension operator in $K_{\mathcal{S}}$. Assuming by contradiction that an extension operator E_F exists, then, for each $\alpha \in L(\omega_1)$ (which is an isolated point in F), let $\xi_\alpha \in C(K_{\mathcal{S}})$ denote the image under E_F of the characteristic function of $\{\alpha\}$. By the continuity of ξ_α, there exists $\phi(\alpha) \in s_\alpha$ with $\xi_\alpha\big(\phi(\alpha)\big) > \frac{1}{2}$. The map $\phi : L(\omega_1) \to \omega_1$ is regressive. Hence the Pressing Down Lemma ([10, Lemma II.6.15]) yields a stationary subset S of $L(\omega_1)$ on which ϕ is constant and equal to some $\beta \in \omega_1$. Take a natural number n greater than $2\|E_F\|$. Since S is infinite, we can pick distinct elements $\alpha_1, \dots, \alpha_n \in S$, obtaining:

$$\|E_F\| \geq \|\xi_{\alpha_1} + \cdots + \xi_{\alpha_n}\| \geq \xi_{\alpha_1}(\beta) + \cdots + \xi_{\alpha_n}(\beta) > \frac{n}{2},$$

a contradiction

A topological space X is said to satisfy the *countable chain condition* (briefly: X is *ccc*) if every family of disjoint nonempty open subsets of X is countable.

Proposition 3.15. *Let K be a compact space and F be a closed subset of K. If K is ccc and F admits an extension operator in K, then F is ccc.*

Proof. Notice that the range of an extension operator $E_F : C(F) \to C(K)$ is an isomorphic copy of $C(F)$ in $C(K)$. The conclusion follows using the fact that a compact space K fails to be ccc if and only if $C(K)$ contains an isomorphic copy of $c_0(I)$ for some uncountable set I (see [6, Theorem 14.26]). □

Corollary 3.16. *If a compact space K has the extension property and is ccc, then K has countable spread, i.e., every discrete subset of K is countable.*

Proof. Notice that, if D is an uncountable discrete subset of K, then $F := \overline{D}$ is not ccc, since the points of D are isolated in F. □

Example 3.17. For an uncountable cardinal κ, the space 2^κ does not have the extension property. Namely, 2^κ is ccc ([10, Theorem II.1.9]) and the characteristic functions of the singleton subsets of κ form a discrete subset of 2^κ of cardinality κ. Hence Corollary 3.16 applies. We can even show that 2^κ is not a continuous image of a compact space having the extension property. Namely, since the ladder system space $K_{\mathcal{S}}$ is a Boolean space of weight ω_1, it is homeomorphic to a subspace of 2^{ω_1} (and hence also of 2^κ). But $K_{\mathcal{S}}$ is not a continuous image of a compact space having the extension property (Proposition 3.10 and Example 3.14).

Example 3.18. Another example of a compact space of height 3 that does not have the extension property is *Mrówka's space* (the one-point compactification of the space constructed in [11]). Namely, Mrówka's space is separable (hence ccc) and has uncountable spread. Then Corollary 3.16 applies.

A Banach space X has the *Grothendieck property* if every operator from X to a separable Banach space is weakly compact.

Proposition 3.19. *Let K be an infinite compact space such that $C(K)$ has the Grothendieck property. Then K does not have the extension property. Moreover, K is not a continuous image of a compact space having the extension property.*

Proof. If K is infinite, then $C(K)$ contains an isometric copy of c_0 (see the proof of [6, Theorem 14.26 (i)]). If K were a continuous image of a compact space having the extension property, then this copy of c_0 would be complemented in $C(K)$ (Remark 2.8). But every separable complemented subspace of a space with the Grothendieck property is reflexive. $\square$

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