

LOCAL EXTENSION PROPERTY FOR FINITE HEIGHT SPACES

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ABSTRACT. We introduce a new technique for the study of the local extension property (LEP) for boolean algebras and we use it to show that the clopen algebra of every compact Hausdorff space K of finite height has LEP. This implies, under appropriate additional assumptions on K and Martin's Axiom, that every twisted sum of c_0 and $C(K)$ is trivial, generalizing a recent result by Marciszewski and Plebanek.

1. INTRODUCTION

The purpose of this article is threefold: first (Section 2), we present a generalization of the construction of Ψ -spaces ([6, 8]) that is relevant for understanding the clopen algebra of a compact Hausdorff scattered space. The second purpose (Section 3) is to introduce a combinatorial method for the study of embeddings of pairs of finite boolean algebras, which is useful to establish that a given boolean algebra has the local extension property (LEP). Finally (Section 4), we use both techniques to prove that the clopen algebra of every compact Hausdorff space of finite height has LEP.

The local extension property for boolean algebras (see Definition 3.2 and Remark 3.3) was introduced by Marciszewski and Plebanek in [7] with the purpose of proving that, under Martin's Axiom (MA), it holds that every twisted sum of c_0 and $C(K)$ is trivial, for certain nonmetrizable compact Hausdorff spaces K . The examples presented in [7] were the first (consistent) counterexamples to the conjecture of Cabello, Castillo, Kalton, and Yost [2, 3, 4, 5] that, for any nonmetrizable compact Hausdorff space K , there exists a *nontrivial twisted sum of c_0 and $C(K)$* , i.e., a Banach space X containing a noncomplemented isomorphic copy Y of c_0 such that X/Y is isomorphic to $C(K)$. As usual, $C(K)$ denotes the Banach space of continuous real-valued functions on K , endowed with the supremum norm.

In [7, Theorem 3.4, Theorem 5.1], the authors show that, under Martin's Axiom, given a separable zero-dimensional compact Hausdorff space K with weight less than the *continuum*, if the algebra $\text{Clop}(K)$ of clopen subsets of K has LEP, then every twisted sum of c_0 and $C(K)$ is trivial. They

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also show that a free boolean algebra (equivalently, the clopen algebra of a power of 2) has LEP ([7, Proposition 4.7]) and that if K is the one-point compactification of a Ψ -space, then $\text{Clop}(K)$ has LEP ([7, Proposition 4.9]).

Recall that a Ψ -space is a separable locally compact Hausdorff space of height 2. The standard construction of a Ψ -space is done using an almost disjoint family $(V_\gamma)_{\gamma \in \Gamma}$ of subsets of ω which defines a topology in the disjoint union $\omega \cup \Gamma$. In this topology, ω is an open dense discrete subspace and the fundamental neighborhoods of each $\gamma \in \Gamma$ are unions of $\{\gamma\}$ with cofinite subsets of V_γ . In Section 2, we generalize this construction by replacing the countable discrete space ω with an arbitrary locally compact Hausdorff space and the almost disjoint family $(V_\gamma)_{\gamma \in \Gamma}$ with an antichain in an appropriate boolean algebra. This generalization allows us to iterate the process in order to produce an arbitrary compact Hausdorff space of finite height, obtaining then a description of its clopen algebra in terms of the various antichains used in the construction. Though this iterative process can be used a transfinite number of times to obtain a similar description for the clopen algebra of an arbitrary compact Hausdorff scattered space, we leave open the problem of whether the clopen algebra has LEP for spaces of infinite height.

2. GENERALIZED Ψ -SPACES

Throughout this section, $\mathcal{X}$ denotes a topological space and Γ denotes an arbitrary set with $\mathcal{X} \cap \Gamma = \emptyset$. We denote by $\wp(\mathcal{X})$ the boolean algebra of all subsets of $\mathcal{X}$, by $\text{cb}(\mathcal{X})$ the boolean subalgebra of $\wp(\mathcal{X})$ consisting of sets with compact boundary and by $\text{rc}(\mathcal{X})$ the ideal of $\text{cb}(\mathcal{X})$ consisting of relatively compact sets, i.e., sets with compact closure. Our initial purpose is to characterize the topologies on $\mathcal{X} \cup \Gamma$ for which $\mathcal{X}$ is an open dense subspace and Γ is a discrete subspace. This characterization is given in terms of families of filters.

Definition 2.1. By the topology on $\mathcal{X} \cup \Gamma$ induced by a given a family $(\mathfrak{F}_\gamma)_{\gamma \in \Gamma}$ of proper filters of $\wp(\mathcal{X})$, we mean the topology in which a subset U of $\mathcal{X} \cup \Gamma$ is open if and only if $U \cap \mathcal{X}$ is open in $\mathcal{X}$ and $U \cap \Gamma$ is in $\mathfrak{F}_\gamma$, for all $\gamma \in \Gamma$.

In order to obtain an one-to-one correspondence between certain topologies on $\mathcal{X} \cup \Gamma$ and families of filters, we need to restrict ourselves to filters satisfying an additional condition.

Definition 2.2. We say that a filter $\mathfrak{F}$ of $\wp(\mathcal{X})$ is *closed under interiors* if for every $V \in \mathfrak{F}$, the interior $\text{int}(V)$ of V belongs to $\mathfrak{F}$.

Proposition 2.3. The topology on $\mathcal{X} \cup \Gamma$ induced by a family of proper filters $(\mathfrak{F}_\gamma)_{\gamma \in \Gamma}$ satisfies the following conditions:

- (a) the subspace topology of $\mathcal{X}$ coincides with the original topology of $\mathcal{X}$;
- (b) $\mathcal{X}$ is open and dense in $\mathcal{X} \cup \Gamma$;
- (c) the subspace topology of Γ is discrete.

If each filter $\mathfrak{F}_\gamma$ is closed under interiors, then:

- (1) $\mathfrak{F}_\gamma = \{U \cap \mathcal{X} : U \text{ is a nhood of } \gamma \text{ in } \mathcal{X} \cup \Gamma\}$ and
- (2) $\mathfrak{F}_\gamma = \{V \in \wp(\mathcal{X}) : V \cup \{\gamma\} \text{ is a nhood of } \gamma \text{ in } \mathcal{X} \cup \Gamma\}.$

Moreover, given a topology on $\mathcal{X} \cup \Gamma$ satisfying (a), (b) and (c), the sets defined in (1) are proper closed under interiors filters of $\wp(\mathcal{X})$ and the given topology on $\mathcal{X} \cup \Gamma$ coincides with the topology induced by this family of filters.

Proof. Assume that $\mathcal{X} \cup \Gamma$ is endowed with the topology induced by a family $(\mathfrak{F}_\gamma)_{\gamma \in \Gamma}$ of proper filters. The fact that this topology satisfies (a), (b) and (c) is straightforward. Assuming that $\mathfrak{F}_\gamma$ is closed under interiors, the proof of (1) and (2) follows by noting that, for $V \in \mathfrak{F}_\gamma$, we have that $\text{int}(V) \cup \{\gamma\}$ is open in $\mathcal{X} \cup \Gamma$ and thus $V \cup \{\gamma\}$ is a neighborhood of γ in $\mathcal{X} \cup \Gamma$. Now assume that $\mathcal{X} \cup \Gamma$ is endowed with a topology τ satisfying (a), (b) and (c) and let $\mathfrak{F}_\gamma$ be defined by (1). The fact that $\mathfrak{F}_\gamma$ is a proper filter closed under interiors and the fact that τ is contained in the topology induced by $(\mathfrak{F}_\gamma)_{\gamma \in \Gamma}$ are immediate. Finally, for U open in the topology induced by $(\mathfrak{F}_\gamma)_{\gamma \in \Gamma}$, use the fact that Γ is discrete in the τ -subspace topology to check that U is a τ -neighborhood of every $\gamma \in U \cap \Gamma$. $\square$

Now we wish to study locally compact Hausdorff topologies on $\mathcal{X} \cup \Gamma$ induced by families of filters. We show below that such topologies can be described in terms of antichains in the quotient algebra $\text{cb}(\mathcal{X})/\text{rc}(\mathcal{X})$. Recall that the topology of a Ψ -space is defined through an antichain in $\wp(\omega)/\text{Fin}(\omega)$, where $\text{Fin}(\omega)$ denotes the ideal of $\wp(\omega)$ consisting of finite sets. The algebra $\text{cb}(\mathcal{X})/\text{rc}(\mathcal{X})$ is the appropriate generalization of $\wp(\omega)/\text{Fin}(\omega)$ when the discrete space ω is replaced with an arbitrary locally compact Hausdorff space $\mathcal{X}$. For any boolean algebra $\mathcal{B}$ and any ideal $\mathcal{J}$ of $\mathcal{B}$, we denote by $[b] \in \mathcal{B}/\mathcal{J}$ the class of an element $b \in \mathcal{B}$.

Proposition 2.4. *Assume that $\mathcal{X}$ is locally compact and Hausdorff. Let $\mathcal{X} \cup \Gamma$ be endowed with the topology induced by a family of proper closed under interiors filters $(\mathfrak{F}_\gamma)_{\gamma \in \Gamma}$. We have that this topology is locally compact Hausdorff if and only if there exists an (automatically unique) antichain $(v_\gamma)_{\gamma \in \Gamma}$ in the quotient algebra $\text{cb}(\mathcal{X})/\text{rc}(\mathcal{X})$ such that v_γ is a basis for the filter $\mathfrak{F}_\gamma$, for every γ .*

Proof. If $\mathcal{X} \cup \Gamma$ is locally compact and Hausdorff, pick a compact neighborhood of each $\gamma \in \Gamma$ of the form $V_\gamma \cup \{\gamma\}$, with V_γ a subset of $\mathcal{X}$. It is easy to see that $V_\gamma \in \text{cb}(\mathcal{X})$ and that $v_\gamma = [V_\gamma]$ is a basis for the filter $\mathfrak{F}_\gamma$. The fact that $\mathcal{X}$ is Hausdorff implies that $(v_\gamma)_{\gamma \in \Gamma}$ is an antichain. Conversely, let $(v_\gamma)_{\gamma \in \Gamma}$ be an antichain in $\text{cb}(\mathcal{X})/\text{rc}(\mathcal{X})$ and assume that v_γ is a basis for $\mathfrak{F}_\gamma$. Note that, if $V \in v_\gamma$, then the union of $\{\gamma\}$ with the closure of V in $\mathcal{X}$ is a compact neighborhood of γ in $\mathcal{X} \cup \Gamma$. Finally, to see that $\mathcal{X} \cup \Gamma$ is Hausdorff, observe that if C is a compact neighborhood of some x in $\mathcal{X}$ and if $V \in v_\gamma$, then $(V \setminus C) \cup \{\gamma\}$ is a neighborhood of γ in $\mathcal{X} \cup \Gamma$ disjoint from C . $\square$

We now focus on the case when $\mathcal{X}$ is zero-dimensional. In this case, it is possible to replace $\text{cb}(\mathcal{X})$ with a simpler boolean algebra. More precisely, we have the following result.

Proposition 2.5. *In the statement of Proposition 2.4, if $\mathcal{X}$ is in addition assumed to be zero-dimensional, then we can replace $\text{cb}(\mathcal{X})/\text{rc}(\mathcal{X})$ with $\text{Clop}(\mathcal{X})/\text{Clop}_c(\mathcal{X})$, where $\text{Clop}_c(\mathcal{X})$ denotes the ideal of $\text{Clop}(\mathcal{X})$ given by:*

$$\text{Clop}_c(\mathcal{X}) = \{C \in \text{Clop}(\mathcal{X}) : C \text{ is compact}\}.$$

Moreover, if the topology on $\mathcal{X} \cup \Gamma$ is locally compact Hausdorff, then it is automatically zero-dimensional.

Proof. If $\mathcal{X}$ is zero-dimensional and locally compact, then for every V in $\text{cb}(\mathcal{X})$, there exists $C \in \text{Clop}_c(\mathcal{X})$ containing the boundary of V in $\mathcal{X}$. Thus, $V \setminus C \in \text{Clop}(\mathcal{X})$ and $[V] = [V \setminus C]$. It follows that the inclusion homomorphism of $\text{Clop}(\mathcal{X})$ into $\text{cb}(\mathcal{X})$ passes to the quotient and yields an isomorphism:

$$i : \text{Clop}(\mathcal{X})/\text{Clop}_c(\mathcal{X}) \longrightarrow \text{cb}(\mathcal{X})/\text{rc}(\mathcal{X}).$$

Moreover, for every $v \in \text{Clop}(\mathcal{X})/\text{Clop}_c(\mathcal{X})$, we have that both v and $i(v)$ are bases of the same filter of $\wp(\mathcal{X})$. $\square$

We are now interested in iterating the process described so far of defining topologies in terms of antichains by adding a new layer of points to $\mathcal{X} \cup \Gamma$. This involves the consideration of antichains in $\text{Clop}(\mathcal{X} \cup \Gamma)/\text{Clop}_c(\mathcal{X} \cup \Gamma)$. It is useful to have a strategy that allows us to work within a fixed boolean algebra as we continue to add layers of points. To this aim, we give the following definition.

Definition 2.6. Given a dense subset D of $\mathcal{X}$, we set:

$$\begin{aligned} \text{Clop}^D(\mathcal{X}) &= \{C \cap D : C \in \text{Clop}(\mathcal{X})\} \quad \text{and} \\ \text{Clop}_c^D(\mathcal{X}) &= \{C \cap D : C \in \text{Clop}_c(\mathcal{X})\}. \end{aligned}$$

Note that $\text{Clop}^D(\mathcal{X})$ is a subalgebra of $\wp(D)$ and that:

$$(3) \quad \text{Clop}(\mathcal{X}) \ni C \longmapsto C \cap D \in \text{Clop}^D(\mathcal{X})$$

is an isomorphism that carries the ideal $\text{Clop}_c(\mathcal{X})$ onto $\text{Clop}_c^D(\mathcal{X})$. Thus (3) induces an isomorphism:

$$(4) \quad \text{Clop}(\mathcal{X})/\text{Clop}_c(\mathcal{X}) \longrightarrow \text{Clop}^D(\mathcal{X})/\text{Clop}_c^D(\mathcal{X}).$$

Definition 2.7. Assume that $\mathcal{X}$ is locally compact Hausdorff and zero-dimensional and let D be a dense subset of $\mathcal{X}$. Given an antichain $(v_\gamma)_{\gamma \in \Gamma}$ in $\text{Clop}^D(\mathcal{X})/\text{Clop}_c^D(\mathcal{X})$, by the topology *induced* on $\mathcal{X} \cup \Gamma$ by this antichain we mean the topology induced by the family of filters $(\mathfrak{F}_\gamma)_{\gamma \in \Gamma}$, where $\mathfrak{F}_\gamma$ is the filter whose basis is the image of v_γ under the inverse of isomorphism (4).

In order to describe $\text{Clop}^D(\mathcal{X} \cup \Gamma)$ and $\text{Clop}_c^D(\mathcal{X} \cup \Gamma)$ in terms of the antichain $(v_\gamma)_{\gamma \in \Gamma}$, it is useful to introduce some terminology in the framework of abstract boolean algebras.

Definition 2.8. Let $\mathcal{B}$ be a boolean algebra and $(b_\gamma)_{\gamma \in \Gamma}$ be an antichain in $\mathcal{B}$. An element b of $\mathcal{B}$ is called a *separator* of $(b_\gamma)_{\gamma \in \Gamma}$ if for each $\gamma \in \Gamma$ we have either $b_\gamma \leq b$ or $b_\gamma \wedge b = 0$. By a *trivial separator* of $(b_\gamma)_{\gamma \in \Gamma}$ we mean an element of $\mathcal{B}$ which is the supremum of a finite subset of $\{b_\gamma : \gamma \in \Gamma\}$.

The separators of an antichain in $\mathcal{B}$ form a subalgebra of $\mathcal{B}$ and the trivial separators form an ideal of this subalgebra. For simplicity, given an ideal $\mathcal{J}$ of $\mathcal{B}$, we say that a family $(b_\gamma)_{\gamma \in \Gamma}$ of elements of $\mathcal{B}$ is an *antichain modulo* $\mathcal{J}$ if $([b_\gamma])_{\gamma \in \Gamma}$ is an antichain in the quotient algebra $\mathcal{B}/\mathcal{J}$. Similarly, we say that an element b of $\mathcal{B}$ is a *separator modulo* $\mathcal{J}$ (resp., *trivial separator modulo* $\mathcal{J}$) of $(b_\gamma)_{\gamma \in \Gamma}$ if $[b]$ is a separator (resp., trivial separator) of $([b_\gamma])_{\gamma \in \Gamma}$.

Proposition 2.9. Assume that $\mathcal{X}$ is locally compact Hausdorff and zero-dimensional. Let D be a dense subset of $\mathcal{X}$ and consider $\mathcal{X} \cup \Gamma$ endowed with the topology induced by an antichain $(v_\gamma)_{\gamma \in \Gamma}$ in $\text{Clop}^D(\mathcal{X})/\text{Clop}_c^D(\mathcal{X})$. Then $\text{Clop}^D(\mathcal{X} \cup \Gamma)$ is the subalgebra of $\text{Clop}^D(\mathcal{X})$ given by:

$$\text{Clop}^D(\mathcal{X} \cup \Gamma) = \{C \in \text{Clop}^D(\mathcal{X}) : [C] \text{ is a separator of } (v_\gamma)_{\gamma \in \Gamma}\}$$

and $\text{Clop}_c^D(\mathcal{X} \cup \Gamma)$ is the ideal of $\text{Clop}^D(\mathcal{X} \cup \Gamma)$ given by:

$$\text{Clop}_c^D(\mathcal{X} \cup \Gamma) = \{C \in \text{Clop}^D(\mathcal{X}) : [C] \text{ is a trivial separator of } (v_\gamma)_{\gamma \in \Gamma}\}.$$

Proof. Let $(\tilde{v}_\gamma)_{\gamma \in \Gamma}$ be the antichain in $\text{Clop}(\mathcal{X})/\text{Clop}_c(\mathcal{X})$ corresponding to $(v_\gamma)_{\gamma \in \Gamma}$ through (4). Note that a subset C of $\mathcal{X} \cup \Gamma$ is clopen if and only if $C \cap \mathcal{X} \in \text{Clop}(\mathcal{X})$, $[C \cap \mathcal{X}] \geq \tilde{v}_\gamma$, for all $\gamma \in C \cap \Gamma$, and $[C \cap \mathcal{X}] \wedge \tilde{v}_\gamma = 0$, for all $\gamma \in \Gamma \setminus C$. Moreover, C is a compact clopen of $\mathcal{X} \cup \Gamma$ if and only if either $C \in \text{Clop}_c(\mathcal{X})$ or C is of the form $F \cup \bigcup_{\gamma \in F} V_\gamma$, with $F \subset \Gamma$ finite and $V_\gamma \in \tilde{v}_\gamma$, for $\gamma \in F$. $\square$

2.1. The clopen algebra of a finite height space. Let K be an infinite compact Hausdorff space of finite height and denote by $K^{(n)}$ its n -th Cantor–Bendixson derivative. Let $N \geq 0$ be such that $K^{(N+1)}$ is finite and nonempty and set:

$$\Gamma_n = K^{(n)} \setminus K^{(n+1)}, \quad n = 0, 1, \dots, N+1,$$

so that $\{\Gamma_n : 0 \leq n \leq N+1\}$ is a partition of K . For each n , we have that the union:

$$\mathcal{X}_n = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_n = K \setminus K^{(n+1)}$$

is open in K and thus locally compact Hausdorff zero-dimensional. Moreover, the subspace topology of each Γ_n is discrete and $\mathcal{X}_0 = \Gamma_0$ is dense in $\mathcal{X}_n$. It follows from Proposition 2.5 that the topology of $\mathcal{X}_{n+1} = \mathcal{X}_n \cup \Gamma_{n+1}$ is induced by an antichain $(v_\gamma^{n+1})_{\gamma \in \Gamma_{n+1}}$ in $\mathcal{A}_n/\mathcal{J}_n$, where:

$$\mathcal{A}_n = \text{Clop}^{\Gamma_0}(\mathcal{X}_n) \quad \text{and} \quad \mathcal{J}_n = \text{Clop}_c^{\Gamma_0}(\mathcal{X}_n).$$

For each $n \geq 1$ and $\gamma \in \Gamma_n$, we pick $V_\gamma^n \in v_\gamma^n$, so that $(V_\gamma^n)_{\gamma \in \Gamma_n}$ is an antichain in $\mathcal{A}_{n-1}$ modulo $\mathcal{J}_{n-1}$. It follows from Proposition 2.9 that:

$$\begin{aligned} \mathcal{A}_n &= \{C \in \mathcal{A}_{n-1} : C \text{ is a separator of } (V_\gamma^n)_{\gamma \in \Gamma_n} \text{ modulo } \mathcal{J}_{n-1}\} \quad \text{and} \\ \mathcal{J}_n &= \{C \in \mathcal{A}_{n-1} : C \text{ is a trivial separator of } (V_\gamma^n)_{\gamma \in \Gamma_n} \text{ modulo } \mathcal{J}_{n-1}\}. \end{aligned}$$

We also set:

$$\mathcal{A}_{-1} = \wp(\Gamma_0), \quad \mathcal{J}_{-1} = \{\emptyset\} \quad \text{and} \quad V_\gamma^0 = \{\gamma\}, \quad \text{for } \gamma \in \Gamma_0.$$

Note that:

$$\mathcal{J}_{-1} \subset \mathcal{J}_0 \subset \cdots \subset \mathcal{J}_{N+1} = \text{Clop}^{\Gamma_0}(K) = \mathcal{A}_{N+1} \subset \cdots \subset \mathcal{A}_0 = \mathcal{A}_{-1}.$$

In the proof of our main result in Section 4, it will be useful to assume that:

$$(5) \quad \Gamma_0 = \bigcup_{\gamma \in \Gamma_{N+1}} V_\gamma^{N+1},$$

which is possible since $\Gamma_0 \in \mathcal{J}_{N+1}$ and thus $[\Gamma_0] = \bigvee_{\gamma \in \Gamma_{N+1}} v_\gamma^{N+1}$.

We conclude this section with a concrete description of the boolean algebra $\text{Clop}^{\Gamma_0}(K)$, which is isomorphic to $\text{Clop}(K)$.

Proposition 2.10. *In the context described above, we have that $\mathcal{J}_n$ is the ideal of $\text{Clop}^{\Gamma_0}(K)$ generated by:*

$$(6) \quad \{V_\gamma^i : \gamma \in \Gamma_i, i = 0, 1, \dots, n\},$$

for all n . Moreover, $\text{Clop}^{\Gamma_0}(K)$ is the subalgebra of $\wp(\Gamma_0)$ generated by:

$$(7) \quad \{V_\gamma^n : \gamma \in \Gamma_n, n = 0, 1, \dots, N+1\}.$$

Proof. The first statement is easily proven by induction on n and, since $\mathcal{J}_{N+1} = \text{Clop}^{\Gamma_0}(K)$, the second follows from the fact that $\mathcal{J}_n$ is contained in the subalgebra of $\wp(\Gamma_0)$ generated by (6), which is also proven by induction on n . $\square$

3. EXTENSION OF COMPATIBLE MEASURES

We start by introducing some notation and recalling some elementary facts. Given a boolean algebra $\mathcal{B}$, we denote by $\text{atom}(\mathcal{B})$ the set of all atoms of $\mathcal{B}$ and by $M(\mathcal{B})$ the Banach space of bounded finitely additive real-valued signed measures on $\mathcal{B}$, endowed with the total variation norm. For any subset S of $\mathcal{B}$, we denote by $\langle S \rangle$ the subalgebra generated by S . If $\mathcal{B}$ is finite, then the atoms of $\mathcal{B}$ form a finite partition of unity and there is a natural isomorphism between $\mathcal{B}$ and $\wp(\text{atom}(\mathcal{B}))$, which associates to each $b \in \mathcal{B}$ the set of atoms of $\mathcal{B}$ below b . Moreover, the map that carries each $\mu \in M(\mathcal{B})$ to its restriction to $\text{atom}(\mathcal{B})$ is a linear isometry from $M(\mathcal{B})$ onto the space $\ell_1(\text{atom}(\mathcal{B}))$ of maps $f : \text{atom}(\mathcal{B}) \rightarrow \mathbb{R}$, endowed with the norm $\|f\| = \sum_{b \in \text{atom}(\mathcal{B})} |f(b)|$.

The set of all finite subalgebras of a boolean algebra, ordered by inclusion, is a lattice. If X is an arbitrary set, then the lattice of finite subalgebras

of $\wp(X)$ is anti-isomorphic to the lattice of all equivalence relations $\sim$ on X for which the quotient $X/\sim$ is finite. This anti-isomorphism associates to each finite boolean subalgebra of $\wp(X)$ the equivalence relation whose equivalence classes are the atoms of the subalgebra. It follows that, given finite subalgebras $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\wp(X)$ corresponding to equivalence relations $\sim_1$ and $\sim_2$, then $\langle \mathcal{B}_1 \cup \mathcal{B}_2 \rangle$ corresponds to the intersection of $\sim_1$ with $\sim_2$ and $\mathcal{B}_1 \cap \mathcal{B}_2$ corresponds to the transitive closure of the union of $\sim_1$ with $\sim_2$.

Definition 3.1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be finite subalgebras of a boolean algebra $\mathcal{B}$ with $\mathcal{B} = \langle \mathcal{B}_1 \cup \mathcal{B}_2 \rangle$. We say that two measures $\mu_1 \in M(\mathcal{B}_1)$ and $\mu_2 \in M(\mathcal{B}_2)$ are *compatible* if they agree on $\mathcal{B}_1 \cap \mathcal{B}_2$. The *extension constant* $c(\mathcal{B}_1, \mathcal{B}_2)$ is defined as the infimum of the set of all $c \geq 0$ such that for every pair of compatible measures $\mu_1 \in M(\mathcal{B}_1)$ and $\mu_2 \in M(\mathcal{B}_2)$ there exists a common extension $\mu \in M(\mathcal{B})$ with:

$$(8) \quad \|\mu\| \leq c(\|\mu_1\| + \|\mu_2\|).$$

It is shown in [1, Lemma 0.1] that two compatible measures always admit a common extension. Moreover, it follows from [1, Theorem 1.5] that the extension constant of any pair of finite subalgebras is finite. The challenge is to find upper bounds for extension constants when we are dealing with an infinite family of finite boolean algebras.

Definition 3.2. Given $r > 0$, we say that a boolean algebra $\mathcal{B}$ has the *local extension property with constant r* , abbreviated as $\text{LEP}(r)$, if there exists a collection Ω of finite subalgebras of $\mathcal{B}$ satisfying the following conditions:

- Ω is cofinal in the lattice of finite subalgebras of $\mathcal{B}$;
- in every uncountable subset of Ω there exists a pair of distinct subalgebras $\mathcal{B}_1$ and $\mathcal{B}_2$ with $c(\mathcal{B}_1, \mathcal{B}_2) \leq r$.

We say that $\mathcal{B}$ has LEP if it has $\text{LEP}(r)$, for some $r > 0$.

Remark 3.3. The definition of $\text{LEP}(r)$ appears in [7] in a slightly different form. Firstly, in [7] the authors consider only rational valued measures. This difference is not relevant since, arguing as in the proof of [7, Lemma A.1 (c)], one can show that if $\mu_1 \in M(\mathcal{B}_1)$ and $\mu_2 \in M(\mathcal{B}_2)$ are rational valued and if c is rational, then the existence of a common extension $\mu \in M(\mathcal{B})$ satisfying (8) is equivalent to the existence of a rational valued common extension μ satisfying (8). Moreover, for rational r , the definition of $\text{LEP}(r)$ given in [7] becomes equivalent to ours by replacing $\|\mu_1\| + \|\mu_2\|$ with $\max\{\|\mu_1\|, \|\mu_2\|\}$ in (8). Therefore, if $\mathcal{B}$ has $\text{LEP}(r)$ according to Definition 3.2 and r is rational, then it has $\text{LEP}(2r)$ in the sense of [7].

In what follows we introduce a visual tool that captures all the relevant information about how two finite boolean algebras are embedded in a larger one. This tool is useful for estimating extension constants.

Definition 3.4. A *point diagram* is a triple $(X; F_1, F_2)$ where F_1 and F_2 are finite sets and X is a subset of $F_1 \times F_2$ with $\pi_1[X] = F_1$ and $\pi_2[X] = F_2$, where π_1 and π_2 denote the projections.

A point diagram $(X; F_1, F_2)$ defines embeddings of $\wp(F_1)$ and $\wp(F_2)$ into $\wp(X)$. More precisely, we have injective homomorphisms of boolean algebras:

$$\iota_1 : \wp(F_1) \longrightarrow \wp(X) \quad \text{and} \quad \iota_2 : \wp(F_2) \longrightarrow \wp(X)$$

given by:

$$\iota_1(A) = \pi_1^{-1}[A] \cap X, \quad A \in \wp(F_1) \quad \text{and} \quad \iota_2(A) = \pi_2^{-1}[A] \cap X, \quad A \in \wp(F_2).$$

The subalgebras $\iota_1[\wp(F_1)]$ and $\iota_2[\wp(F_2)]$ correspond, respectively, to the equivalence relations $\sim_1$ and $\sim_2$ on X defined by:

$$(9) \quad x \sim_1 y \iff \pi_1(x) = \pi_1(y) \quad \text{and} \quad x \sim_2 y \iff \pi_2(x) = \pi_2(y),$$

for all $x, y \in X$. It follows that the copies of $\wp(F_1)$ and $\wp(F_2)$ in $\wp(X)$ generate $\wp(X)$, i.e.:

$$\wp(X) = \langle \iota_1[\wp(F_1)] \cup \iota_2[\wp(F_2)] \rangle.$$

Conversely, any pair of embeddings of finite boolean algebras defines a point diagram as explained in next definition.

Definition 3.5. Given finite subalgebras $\mathcal{B}_1$ and $\mathcal{B}_2$ of a boolean algebra $\mathcal{B}$, the *point diagram of $\mathcal{B}_1$ and $\mathcal{B}_2$* is the triple $(X; F_1, F_2)$ defined by setting $F_1 = \text{atom}(\mathcal{B}_1)$, $F_2 = \text{atom}(\mathcal{B}_2)$ and:

$$X = \{(a_1, a_2) \in \text{atom}(\mathcal{B}_1) \times \text{atom}(\mathcal{B}_2) : a_1 \wedge a_2 \neq 0\}.$$

Note that, if $\mathcal{B} = \langle \mathcal{B}_1 \cup \mathcal{B}_2 \rangle$, then we have a bijection:

$$X \ni (a_1, a_2) \longmapsto a_1 \wedge a_2 \in \text{atom}(\mathcal{B})$$

that induces an isomorphism between $\wp(X)$ and $\mathcal{B}$. Moreover, the triples $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ and $(\wp(X), \wp(F_1), \wp(F_2))$ are isomorphic in the sense that the diagrams:

$$\begin{array}{ccc} \wp(X) & \xrightarrow{\cong} & \mathcal{B} \\ \uparrow \iota_1 & & \uparrow \\ \wp(F_1) & \xrightarrow{\cong} & \mathcal{B}_1 \end{array} \quad \begin{array}{ccc} \wp(X) & \xrightarrow{\cong} & \mathcal{B} \\ \uparrow \iota_2 & & \uparrow \\ \wp(F_2) & \xrightarrow{\cong} & \mathcal{B}_2 \end{array}$$

are commutative. Finally, note that any point diagram $(X; F_1, F_2)$ can be naturally identified with the point diagram of the subalgebras $\iota_1[\wp(F_1)]$ and $\iota_2[\wp(F_2)]$ of $\wp(X)$.

Now that we have established an equivalence between embeddings of pairs of finite boolean algebras and point diagrams, we are going to translate the problem of extending compatible measures to the framework of point diagrams.

Definition 3.6. Let $(X; F_1, F_2)$ be a point diagram. We say that a pair of maps $f_1 \in \ell_1(F_1)$ and $f_2 \in \ell_1(F_2)$ is *compatible* if the corresponding measures μ_1 and μ_2 are compatible, i.e., if for every $A_1 \subset F_1$ and every $A_2 \subset F_2$ with:

$$\pi_1^{-1}[A_1] \cap X = \pi_2^{-1}[A_2] \cap X$$

we have $\sum_{a \in A_1} f_1(a) = \sum_{a \in A_2} f_2(a)$. By a *common extension* of f_1 and f_2 , we mean a map $g \in \ell_1(X)$ whose corresponding measure is a common extension of μ_1 and μ_2 , i.e.:

$$f_1(a_1) = \sum_{\substack{a_2 \in F_2 \\ (a_1, a_2) \in X}} g(a_1, a_2), \quad a_1 \in F_1 \quad \text{and} \quad f_2(a_2) = \sum_{\substack{a_1 \in F_1 \\ (a_1, a_2) \in X}} g(a_1, a_2), \quad a_2 \in F_2.$$

Finally, we define the *extension constant* $c(X; F_1, F_2)$ of the point diagram $(X; F_1, F_2)$ to be the infimum of the set of all $c \geq 0$ such that every compatible pair $f_1 \in \ell_1(F_1)$, $f_2 \in \ell_1(F_2)$ admits a common extension $g \in \ell_1(X)$ with $\|g\| \leq c(\|f_1\| + \|f_2\|)$.

Clearly the extension constant of the point diagram $(X; F_1, F_2)$ coincides with the extension constant of the pair of subalgebras $\iota_1[\wp(F_1)]$ and $\iota_2[\wp(F_2)]$ of $\wp(X)$.

Definition 3.7. Given a point diagram $(X; F_1, F_2)$, we call the transitive closure of the union of the equivalence relations $\sim_1$ and $\sim_2$ defined in (9) the *rook* equivalence relation on X .

Note that the equivalence classes of the rook relation are precisely the atoms of the intersection $\iota_1[\wp(F_1)] \cap \iota_2[\wp(F_2)]$. Elements of this intersection can be used to decompose a point diagram as explained below.

Definition 3.8. A *decomposition* of a point diagram $(X; F_1, F_2)$ is a triple of partitions:

$$(10) \quad F_1 = F_1^1 \cup \dots \cup F_1^k, \quad F_2 = F_2^1 \cup \dots \cup F_2^k, \quad X = X^1 \cup \dots \cup X^k$$

with $X^i \subset F_1^i \times F_2^i$, for $i = 1, \dots, k$. The point diagram is called *decomposable* if it admits a nontrivial decomposition, i.e., a decomposition with $k \geq 2$ and $X^i \neq \emptyset$, $i = 1, \dots, k$. Otherwise, it is called *indecomposable*.

Note that each $(X^i; F_1^i, F_2^i)$ is itself a point diagram and that:

$$X^i = \iota_1(F_1^i) = \iota_2(F_2^i),$$

so that X^i belongs to the intersection $\iota_1[\wp(F_1)] \cap \iota_2[\wp(F_2)]$, for all $i = 1, \dots, k$. Conversely, if $X = X^1 \cup \dots \cup X^k$ is a partition of X into elements of the intersection $\iota_1[\wp(F_1)] \cap \iota_2[\wp(F_2)]$, then we obtain a decomposition of $(X; F_1, F_2)$ into point diagrams $(X^i; F_1^i, F_2^i)$ by setting $F_1^i = \iota_1^{-1}(X^i)$ and $F_2^i = \iota_2^{-1}(X^i)$. It follows that a point diagram $(X; F_1, F_2)$ is indecomposable if and only if the intersection $\iota_1[\wp(F_1)] \cap \iota_2[\wp(F_2)]$ is trivial. Note also that, for an indecomposable point diagram $(X; F_1, F_2)$, a pair of maps $f_1 \in \ell_1(F_1)$, $f_2 \in \ell_1(F_2)$ is compatible if and only if $\sum_{a \in F_1} f_1(a) = \sum_{a \in F_2} f_2(a)$.

Lemma 3.9. *If (10) is a decomposition of a point diagram $(X; F_1, F_2)$, then:*

$$c(X; F_1, F_2) = \max_{1 \leq i \leq k} c(X^i; F_1^i, F_2^i).$$

Proof. Note that $f_1 \in \ell_1(F_1)$ and $f_2 \in \ell_1(F_2)$ are compatible if and only if $f_1|_{F_1^i}$ and $f_2|_{F_2^i}$ are compatible for all i . Moreover, $g \in \ell_1(X)$ is a common extension of f_1 and f_2 if and only if $g|_{X^i}$ is a common extension of $f_1|_{F_1^i}$ and $f_2|_{F_2^i}$ for all i . $\square$

Lemma 3.10. *Let $(X; F_1, F_2)$ be a point diagram. Assume that the sets F_1 and F_2 are written as disjoint unions:*

$$F_1 = I_1 \cup J_1 \quad \text{and} \quad F_2 = I_2 \cup J_2$$

such that the following conditions are satisfied:

- (i) $(X \cap (J_1 \times J_2); J_1, J_2)$ is an indecomposable point diagram;
- (ii) for all $a_1 \in I_1$, there exists $a_2 \in J_2$ with $(a_1, a_2) \in X$;
- (iii) for all $a_2 \in I_2$, there exists $a_1 \in J_1$ with $(a_1, a_2) \in X$.

Then $(X; F_1, F_2)$ is indecomposable and:

$$(11) \quad c(X; F_1, F_2) \leq c(X \cap (J_1 \times J_2); J_1, J_2) + 1.$$

Proof. To see that $(X; F_1, F_2)$ is indecomposable, note that every element of X is rook equivalent to an element of $X \cap (J_1 \times J_2)$, so that the only rook equivalence class is X itself. Now let us prove (11). Let $f_1 \in \ell_1(F_1)$ and $f_2 \in \ell_1(F_2)$ be compatible. By conditions (ii) and (iii), there exist functions $\phi : I_1 \rightarrow J_2$ and $\psi : I_2 \rightarrow J_1$ such that:

$$(a_1, \phi(a_1)) \in X \quad \text{and} \quad (\psi(a_2), a_2) \in X,$$

for all $a_1 \in I_1$ and $a_2 \in I_2$. We define $\tilde{f}_1 \in \ell_1(J_1)$ and $\tilde{f}_2 \in \ell_1(J_2)$ by setting:

$$\begin{aligned} \tilde{f}_1(a_1) &= f_1(a_1) - \sum_{a_2 \in \psi^{-1}(a_1)} f_2(a_2), \quad a_1 \in J_1 \quad \text{and} \\ \tilde{f}_2(a_2) &= f_2(a_2) - \sum_{a_1 \in \phi^{-1}(a_2)} f_1(a_1), \quad a_2 \in J_2. \end{aligned}$$

One readily checks that:

$$\sum_{a_1 \in J_1} \tilde{f}_1(a_1) = \sum_{a_2 \in J_2} \tilde{f}_2(a_2)$$

so that, by (i), $\tilde{f}_1$ and $\tilde{f}_2$ are compatible. Thus, there exists a common extension $\tilde{g} \in \ell_1(X \cap (J_1 \times J_2))$ of $\tilde{f}_1$ and $\tilde{f}_2$ such that:

$$\|\tilde{g}\| \leq c(X \cap (J_1 \times J_2); J_1, J_2)(\|\tilde{f}_1\| + \|\tilde{f}_2\|).$$

A common extension $g \in \ell_1(X)$ of f_1 and f_2 is now obtained by setting $g(a_1, a_2) = f_1(a_1)$, for $a_1 \in I_1$ and $a_2 = \phi(a_1)$, $g(a_1, a_2) = f_2(a_2)$, for $a_2 \in I_2$

and $a_1 = \psi(a_2)$, $g(a_1, a_2) = \tilde{g}(a_1, a_2)$, for $(a_1, a_2) \in X \cap (J_1 \times J_2)$ and $g(a_1, a_2) = 0$, otherwise. A straightforward computation shows that:

$$\|g\| \leq [c(X \cap (J_1 \times J_2); J_1, J_2) + 1](\|f_1\| + \|f_2\|). \quad \square$$

Definition 3.11. Given a nonnegative integer k , we say that a point diagram $(X; F_1, F_2)$ is of type k if the sets F_1 and F_2 can be written as disjoint unions:

$$F_1 = F_1(0) \cup \dots \cup F_1(k) \quad \text{and} \quad F_2 = F_2(0) \cup \dots \cup F_2(k)$$

such that the following conditions hold:

- (a) $F_1(k)$ and $F_2(k)$ are singletons;
- (b) $F_1(k) \times F_2(k) \subset X$;
- (c) for every $i = 0, \dots, k-1$ and every $a_1 \in F_1(i)$, there exists a_2 in $\bigcup_{j=i+1}^k F_2(j)$ with $(a_1, a_2) \in X$;
- (d) for every $i = 0, \dots, k-1$ and every $a_2 \in F_2(i)$, there exists a_1 in $\bigcup_{j=i+1}^k F_1(j)$ with $(a_1, a_2) \in X$.

Lemma 3.12. *If $(X; F_1, F_2)$ is a point diagram of type k , then it is indecomposable and $c(X; F_1, F_2) \leq k + \frac{1}{2}$.*

Proof. We proceed by induction on k . The case $k = 0$ is trivial. If $(X; F_1, F_2)$ is a point diagram of type k , with $k \geq 1$, then $(X \cap (J_1 \times J_2); J_1, J_2)$ is a point diagram of type $k-1$, where $J_1 = \bigcup_{i=1}^k F_1(i)$ and $J_2 = \bigcup_{i=1}^k F_2(i)$. To conclude the proof, set $I_1 = F_1(0)$, $I_2 = F_2(0)$ and apply Lemma 3.10. $\square$

4. MAIN RESULTS

The goal of this section is to prove the following result.

Theorem 4.1. *If K is a compact Hausdorff space with finite height M , then the boolean algebra $\text{Clop}(K)$ has $\text{LEP}(M - \frac{1}{2})$.*

This immediately implies the following generalization of [7, Corollary 5.3].

Corollary 4.2. *Let $\kappa < \mathfrak{c}$ and assume that $\text{MA}(\kappa)$ holds. If K is a separable compact Hausdorff space with finite height and weight less than or equal to κ , then every twisted sum of c_0 and $C(K)$ is trivial.*

Proof. It follows from Theorem 4.1, [7, Theorem 5.1] and [7, Theorem 3.4]. $\square$

To prove Theorem 4.1, we go back to the setup of Subsection 2.1 and we set $\mathcal{B} = \text{Clop}^{\Gamma_0}(K)$. Recall that $\mathcal{B}$ is isomorphic to $\text{Clop}(K)$ and thus, it is sufficient to prove that $\mathcal{B}$ has $\text{LEP}(M - \frac{1}{2})$. Our first step is to exhibit a suitable cofinal subset of the lattice of finite subalgebras of $\mathcal{B}$. We need some notation and terminology. Set:

$$\mathbb{G} = \{(G_n)_{0 \leq n \leq N+1} : G_n \text{ is a finite subset of } \Gamma_n, n = 0, 1, \dots, N+1\}$$

and for every $\mathcal{G} = (G_n)_{0 \leq n \leq N+1} \in \mathbb{G}$ and every subset S of $\{0, 1, \dots, N+1\}$ denote by $V_{\mathcal{G}}^S$ the union:

$$V_{\mathcal{G}}^S = \bigcup \{V_{\gamma}^n : \gamma \in G_n, n \in S\}.$$

Each sequence $\mathcal{G} = (G_n)_{0 \leq n \leq N+1}$ in $\mathbb{G}$ defines a finite subalgebra:

$$\mathcal{B}_{\mathcal{G}} = \langle \{V_{\gamma}^n : \gamma \in G_n, n = 0, 1, \dots, N+1\} \rangle$$

of $\mathcal{B}$ whose set of atoms we denote by $F_{\mathcal{G}}$.

The cofinal subset of finite subalgebras of $\mathcal{B}$ will consist of those $\mathcal{B}_{\mathcal{G}}$ such that the fact that $(V_{\gamma}^n)_{\gamma \in G_n}$ is an antichain in $\mathcal{A}_{n-1}$ modulo $\mathcal{J}_{n-1}$ is witnessed by $\mathcal{G}$. More precisely, we give the following definition.

Definition 4.3. We say that $\mathcal{G} = (G_n)_{0 \leq n \leq N+1} \in \mathbb{G}$ is *admissible* if the following conditions hold:

- (a) $V_{\gamma}^n \cap V_{\delta}^n \subset V_{\mathcal{G}}^{[0, n]}$, for all $n = 1, \dots, N+1$ and all $\gamma, \delta \in G_n$ with $\gamma \neq \delta$;
- (b) either $V_{\gamma}^m \setminus V_{\delta}^n$ is contained in $V_{\mathcal{G}}^{[0, m]}$ or $V_{\gamma}^m \cap V_{\delta}^n$ is contained in $V_{\mathcal{G}}^{[0, m]}$, for all $m, n = 1, \dots, N+1$ with $m < n$ and all $\gamma \in G_m$ and $\delta \in G_n$;
- (c) $G_{N+1} = \Gamma_{N+1}$.

Proposition 4.4. *The collection:*

$$(12) \quad \{\mathcal{B}_{\mathcal{G}} : \mathcal{G} \in \mathbb{G} \text{ admissible}\}$$

is cofinal in the lattice of finite subalgebras of $\mathcal{B}$.

Proof. Since the set (7) generates $\mathcal{B}$, it is sufficient to show that for every $\mathcal{G} = (G_n)_{0 \leq n \leq N+1} \in \mathbb{G}$, there exists an admissible $\mathcal{H} = (H_n)_{0 \leq n \leq N+1} \in \mathbb{G}$ such that G_n is contained in H_n , for all n . The sets H_n are easily constructed by recursion on n , starting at $n = N+1$. $\square$

Our next step is to obtain an explicit description of the atoms of $\mathcal{B}_{\mathcal{G}}$. It follows easily from (5) that, given an admissible $\mathcal{G} = (G_n)_{0 \leq n \leq N+1} \in \mathbb{G}$, the sets:

$$A_{\mathcal{G}}^{n, \gamma} = V_{\gamma}^n \setminus V_{\mathcal{G}}^{[0, n]}, \quad \gamma \in G_n, \quad 0 \leq n \leq N+1$$

form a partition of Γ_0 into elements of $\mathcal{B}_{\mathcal{G}}$. To prove that those are precisely the atoms of $\mathcal{B}_{\mathcal{G}}$, we need an auxiliary result.

Lemma 4.5. *Let X be a set and $C_1, C_2, \dots, C_n$ be subsets of X such that, for $1 \leq i < j \leq n$, either $C_i \cap C_j$ or $C_i \setminus C_j$ is contained in $\bigcup_{k < i} C_k$. Then the subalgebra of $\wp(X)$ generated by $\{C_i : i = 1, 2, \dots, n\}$ coincides with the subalgebra generated by:*

$$(13) \quad \{C_i \setminus \bigcup_{k < i} C_k : i = 1, 2, \dots, n\}.$$

Proof. The proof is done by induction on n . Given $n > 1$ and assuming that the result holds for $n - 1$, we have to check that C_n belongs to the subalgebra generated by (13). To this aim, it suffices to show that $C_n \cap C_i$ belongs to such subalgebra for all $i < n$, which is easily done by induction on i . $\square$

Corollary 4.6. *If $\mathcal{G}$ is admissible, then the atoms of $\mathcal{B}_{\mathcal{G}}$ are:*

$$F_{\mathcal{G}} = \{A_{\mathcal{G}}^{n,\gamma} : \gamma \in G_n, 0 \leq n \leq N + 1\}.$$

Proof. Follows from Lemma 4.5 and the fact that if a subalgebra is generated by a partition, then the elements of the partition are the atoms of this subalgebra. $\square$

At this point, we need to estimate the extension constant for pairs of elements of the cofinal set (12).

Theorem 4.7. *For every admissible sequences $\mathcal{G}, \mathcal{H} \in \mathbb{G}$, we have that the extension constant $c(\mathcal{B}_{\mathcal{G}}, \mathcal{B}_{\mathcal{H}})$ is less than or equal to $N + \frac{3}{2}$.*

We can now use Theorem 4.7 to conclude the proof of Theorem 4.1 and then the reminder of the section will be dedicated to the proof of Theorem 4.7.

Proof of Theorem 4.1. Since $\text{Clop}(K)$ is isomorphic to $\mathcal{B}$ and the height M of K is equal to $N + 2$, the result follows directly from Proposition 4.4 and Theorem 4.7. $\square$

From now on, we consider fixed admissible sequences $\mathcal{G} = (G_n)_{0 \leq n \leq N+1}$ and $\mathcal{H} = (H_n)_{0 \leq n \leq N+1}$ in $\mathbb{G}$ and we set:

$$\mathcal{G} \cap \mathcal{H} = (G_n \cap H_n)_{0 \leq n \leq N+1}.$$

Recall from Section 3 that the pair of subalgebras $\mathcal{B}_{\mathcal{G}}, \mathcal{B}_{\mathcal{H}}$ defines a point diagram $(X; F_{\mathcal{G}}, F_{\mathcal{H}})$. Our strategy is to use this point diagram to estimate the extension constant $c(\mathcal{B}_{\mathcal{G}}, \mathcal{B}_{\mathcal{H}})$. To this aim, we will exhibit a partition of Γ_0 into elements of $\mathcal{B}_{\mathcal{G}} \cap \mathcal{B}_{\mathcal{H}}$ which will yield a decomposition of $(X; F_{\mathcal{G}}, F_{\mathcal{H}})$.

Next lemma states that $\mathcal{G} \cap \mathcal{H}$ witness the fact that $(V_{\gamma}^n)_{\gamma \in G_n \cap H_n}$ is an antichain modulo $\mathcal{J}_{n-1}$.

Lemma 4.8. *For any $n = 1, \dots, N + 1$ and any $\gamma, \delta \in G_n \cap H_n$ with $\gamma \neq \delta$, we have $V_{\gamma}^n \cap V_{\delta}^n \subset V_{\mathcal{G} \cap \mathcal{H}}^{[0,n]}$.*

Proof. Since $\mathcal{G}$ and $\mathcal{H}$ are admissible, we have:

$$(14) \quad V_{\gamma}^n \cap V_{\delta}^n \subset V_{\mathcal{G}}^{[0,n]} \quad \text{and} \quad V_{\gamma}^n \cap V_{\delta}^n \subset V_{\mathcal{H}}^{[0,n]}.$$

To prove the lemma, we will show by induction on m that:

$$(15) \quad V_{\gamma}^n \cap V_{\delta}^n \subset V_{\mathcal{G}}^{[0,m]} \cup V_{\mathcal{G} \cap \mathcal{H}}^{[m,n]} \quad \text{and}$$

$$(16) \quad V_{\gamma}^n \cap V_{\delta}^n \subset V_{\mathcal{H}}^{[0,m]} \cup V_{\mathcal{G} \cap \mathcal{H}}^{[m,n]},$$

for $m = 0, \dots, n$. The case $m = n$ is just (14). Fix $m < n$ and assume that:

$$(17) \quad V_\gamma^n \cap V_\delta^n \subset V_{\mathcal{G}}^{[0,m]} \cup V_{\mathcal{G} \cap \mathcal{H}}^{[m,n[} \quad \text{and}$$

$$(18) \quad V_\gamma^n \cap V_\delta^n \subset V_{\mathcal{H}}^{[0,m]} \cup V_{\mathcal{G} \cap \mathcal{H}}^{[m,n[}.$$

Let us prove (15). The proof of (16) is analogous. It is sufficient to check that:

$$(19) \quad V_\gamma^n \cap V_\delta^n \cap (V_\epsilon^m \setminus V_{\mathcal{G}}^{[0,m]}) \subset V_{\mathcal{G} \cap \mathcal{H}}^{[m,n[},$$

for all $\epsilon \in G_m \setminus H_m$. Since $\mathcal{G}$ is admissible, we have that $A_{\mathcal{G}}^{m,\epsilon} = V_\epsilon^m \setminus V_{\mathcal{G}}^{[0,m]}$ is either contained in V_γ^n or is disjoint from V_γ^n . Similarly, $A_{\mathcal{G}}^{m,\epsilon}$ is either contained in V_δ^n or is disjoint from V_δ^n . If $A_{\mathcal{G}}^{m,\epsilon}$ is disjoint from either V_γ^n or V_δ^n , then (19) holds trivially. Otherwise, it follows from (18) that:

$$(20) \quad A_{\mathcal{G}}^{m,\epsilon} \subset V_{\mathcal{H}}^{[0,m]} \cup V_{\mathcal{G} \cap \mathcal{H}}^{[m,n]}.$$

Since $A_{\mathcal{G}}^{m,\epsilon}$ is an atom of $\mathcal{B}_{\mathcal{G}}$ (Corollary 4.6) and $V_{\mathcal{G} \cap \mathcal{H}}^{[m,n]} \in \mathcal{B}_{\mathcal{G}}$, it follows that $A_{\mathcal{G}}^{m,\epsilon}$ is either contained in $V_{\mathcal{G} \cap \mathcal{H}}^{[m,n]}$ or is disjoint from $V_{\mathcal{G} \cap \mathcal{H}}^{[m,n]}$. If it is contained, then (19) follows. If $A_{\mathcal{G}}^{m,\epsilon}$ is disjoint from $V_{\mathcal{G} \cap \mathcal{H}}^{[m,n]}$ then, by (20), $A_{\mathcal{G}}^{m,\epsilon}$ is contained in $V_{\mathcal{H}}^{[0,m]}$. Therefore:

$$(21) \quad A_{\mathcal{G}}^{m,\epsilon} = A_{\mathcal{G}}^{m,\epsilon} \cap V_{\mathcal{H}}^{[0,m]} \subset V_{\mathcal{H}}^{[0,m]} \cup (V_\epsilon^m \cap V_{\mathcal{H}}^{\{m\}}).$$

Since $\epsilon \notin H_m$, we have that $V_\epsilon^m \cap V_{\mathcal{H}}^{\{m\}} \in \mathcal{J}_{m-1}$ and since $V_{\mathcal{H}}^{[0,m]}$ is also in $\mathcal{J}_{m-1}$, we obtain that (21) contradicts the fact that $A_{\mathcal{G}}^{m,\epsilon}$ is not in $\mathcal{J}_{m-1}$. $\square$

Corollary 4.9. *The sets:*

$$A_{\mathcal{G} \cap \mathcal{H}}^{n,\gamma} = V_\gamma^n \setminus V_{\mathcal{G} \cap \mathcal{H}}^{[0,n]}, \quad \gamma \in G_n \cap H_n, \quad 0 \leq n \leq N+1$$

form a partition of Γ_0 into elements of $\mathcal{B}_{\mathcal{G}} \cap \mathcal{B}_{\mathcal{H}}$. $\square$

The partition of Γ_0 given in Corollary 4.9 induces a partition of $F_{\mathcal{G}}$:

$$\{F_{\mathcal{G}}^{n,\gamma} : \gamma \in G_n \cap H_n, \quad 0 \leq n \leq N+1\},$$

where $F_{\mathcal{G}}^{n,\gamma}$ is the collection of atoms of $\mathcal{B}_{\mathcal{G}}$ contained in $A_{\mathcal{G} \cap \mathcal{H}}^{n,\gamma}$. Similarly, we obtain a partition of $F_{\mathcal{H}}$ into sets $F_{\mathcal{H}}^{n,\gamma}$. Next lemma gives an explicit description of $F_{\mathcal{G}}^{n,\gamma}$.

Lemma 4.10. *For all $n = 0, 1, \dots, N+1$ and all $\gamma \in G_n \cap H_n$, we have that $F_{\mathcal{G}}^{n,\gamma} = \bigcup_{m=0}^n F_{\mathcal{G}}^{n,\gamma}(m)$, where $F_{\mathcal{G}}^{n,\gamma}(n) = \{A_{\mathcal{G}}^{n,\gamma}\}$ and, for $m < n$, the elements of $F_{\mathcal{G}}^{n,\gamma}(m)$ are the atoms $A_{\mathcal{G}}^{m,\delta}$ such that $\delta \in G_m \setminus H_m$, $A_{\mathcal{G}}^{m,\delta} \subset V_\gamma^n$ and $A_{\mathcal{G}}^{m,\delta} \cap V_{\mathcal{G} \cap \mathcal{H}}^{[m,n]} = \emptyset$.*

Proof. Fix n and m with $0 \leq n, m \leq N+1$, $\gamma \in G_n \cap H_n$ and $\delta \in G_m$. Since both V_γ^n and $V_{\mathcal{G} \cap \mathcal{H}}^{[0,n]}$ belong to $\mathcal{B}_{\mathcal{G}}$, we have that the atom $A_{\mathcal{G}}^{m,\delta}$ of $\mathcal{B}_{\mathcal{G}}$ is contained in $A_{\mathcal{G} \cap \mathcal{H}}^{n,\gamma} = V_\gamma^n \setminus V_{\mathcal{G} \cap \mathcal{H}}^{[0,n]}$ if and only if it is contained in V_γ^n and it is disjoint from $V_{\mathcal{G} \cap \mathcal{H}}^{[0,n]}$. If either $m > n$ or if $m = n$ and $\delta \neq \gamma$, then $A_{\mathcal{G}}^{m,\delta}$ is

disjoint from V_γ^n . Clearly, for $m = n$ and $\delta = \gamma$, the set $A_{\mathcal{G}}^{m,\delta}$ is contained in $A_{\mathcal{G} \cap \mathcal{H}}^{n,\gamma}$. Finally, for $m < n$, we have that $A_{\mathcal{G}}^{m,\delta}$ is disjoint from $V_{\mathcal{G} \cap \mathcal{H}}^{[0,n]}$ if and only if $\delta \notin H_m$ and $A_{\mathcal{G}}^{m,\delta} \cap V_{\mathcal{G} \cap \mathcal{H}}^{[m,n]} = \emptyset$. $\square$

Obviously, we have $F_{\mathcal{H}}^{n,\gamma} = \bigcup_{m=0}^n F_{\mathcal{H}}^{n,\gamma}(m)$, with $F_{\mathcal{H}}^{n,\gamma}(m)$ defined as in the statement of Lemma 4.10, interchanging the roles of $\mathcal{G}$ and $\mathcal{H}$.

The partitions of $F_{\mathcal{G}}$ and $F_{\mathcal{H}}$ obtained above yield a decomposition of the point diagram $(X; F_{\mathcal{G}}, F_{\mathcal{H}})$ of $\mathcal{B}_{\mathcal{G}}$ and $\mathcal{B}_{\mathcal{H}}$ by setting:

$$X^{n,\gamma} = X \cap (F_{\mathcal{G}}^{n,\gamma} \times F_{\mathcal{H}}^{n,\gamma}), \quad \gamma \in G_n \cap H_n, \quad 0 \leq n \leq N+1.$$

Lemma 4.11. *For all $n = 0, \dots, N+1$ and all $\gamma \in G_n \cap H_n$, the point diagram $(X^{n,\gamma}; F_{\mathcal{G}}^{n,\gamma}, F_{\mathcal{H}}^{n,\gamma})$ is of type n .*

Proof. Let n with $0 \leq n \leq N+1$ and $\gamma \in G_n \cap H_n$ be fixed. To prove that $(X^{n,\gamma}; F_{\mathcal{G}}^{n,\gamma}, F_{\mathcal{H}}^{n,\gamma})$ is of type n , we consider the disjoint unions:

$$F_{\mathcal{G}}^{n,\gamma} = \bigcup_{m=0}^n F_{\mathcal{G}}^{n,\gamma}(m) \quad \text{and} \quad F_{\mathcal{H}}^{n,\gamma} = \bigcup_{m=0}^n F_{\mathcal{H}}^{n,\gamma}(m).$$

It is clear that condition (a) of Definition 3.11 is satisfied. To prove condition (b), we have to check that $A_{\mathcal{G}}^{n,\gamma}$ intersects $A_{\mathcal{H}}^{n,\gamma}$. If $A_{\mathcal{G}}^{n,\gamma} \cap A_{\mathcal{H}}^{n,\gamma} = \emptyset$, then $A_{\mathcal{G}}^{n,\gamma}$ is contained in:

$$(22) \quad \bigcup \{A : A \in \bigcup_{m < n} F_{\mathcal{H}}^{n,\gamma}(m)\},$$

which yields a contradiction since the union (22) belongs to the ideal $\mathcal{J}_{n-1}$ and $A_{\mathcal{G}}^{n,\gamma}$ does not. Now, let us prove condition (c). Fix $m < n$ and an element $A_{\mathcal{G}}^{m,\delta}$ of $F_{\mathcal{G}}^{n,\gamma}(m)$. We need to show that $A_{\mathcal{G}}^{m,\delta}$ intersects the union:

$$\bigcup \{A : A \in \bigcup_{i > m} F_{\mathcal{H}}^{n,\gamma}(i)\}.$$

Assuming that this does not hold, we have that $A_{\mathcal{G}}^{m,\delta}$ is contained in:

$$\bigcup \{A : A \in \bigcup_{i \leq m} F_{\mathcal{H}}^{n,\gamma}(i)\}.$$

This implies that $A_{\mathcal{G}}^{m,\delta}$ is contained in the union of:

$$(23) \quad \bigcup \{A : A \in \bigcup_{i < m} F_{\mathcal{H}}^{n,\gamma}(i)\}$$

with

$$(24) \quad \bigcup \{A_{\mathcal{G}}^{m,\delta} \cap A_{\mathcal{H}}^{m,\epsilon} : \epsilon \in H_m \setminus G_m\}.$$

Clearly (23) belongs to $\mathcal{J}_{m-1}$. Moreover, since δ is in $G_m \setminus H_m$ and thus $\delta \neq \epsilon$ for all $\epsilon \in H_m \setminus G_m$, we have that also (24) belongs to $\mathcal{J}_{m-1}$. The fact that $A_{\mathcal{G}}^{m,\delta}$ is contained in the union of (23) with (24) therefore contradicts the fact that $A_{\mathcal{G}}^{m,\delta}$ is not in $\mathcal{J}_{m-1}$ and concludes the proof of condition (c). The proof of condition (d) is analogous. $\square$

Proof of Theorem 4.7. Using Lemmas 4.11 and 3.12 we obtain that:

$$c(X^{n,\gamma}; F_{\mathcal{G}}^{n,\gamma}, F_{\mathcal{H}}^{n,\gamma}) \leq n + \frac{1}{2}.$$

The conclusion follows from Lemma 3.9. $\square$

Remark 4.12. In Theorem 4.1, if K is in addition assumed to be separable, then we can improve the constant for the local extension property of $\text{Clop}(K)$. Namely, we can obtain that $\text{Clop}(K)$ has LEP $(M - \frac{3}{2})$. To see this, note that if K is separable, then Γ_0 is countable and thus every uncountable subset of (12) contains a pair of distinct algebras $\mathcal{B}_{\mathcal{G}}$ and $\mathcal{B}_{\mathcal{H}}$ with $G_0 = H_0$. Then it follows from Lemma 4.10 that $F_{\mathcal{G}}^{n,\gamma}(0)$ and $F_{\mathcal{H}}^{n,\gamma}(0)$ are empty for $n > 0$ and hence the proof of Lemma 4.11 yields that the point diagram $(X^{n,\gamma}; F_{\mathcal{G}}^{n,\gamma}, F_{\mathcal{H}}^{n,\gamma})$ has type $n - 1$.

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