A NOTE ON THE CONTINUOUS SELF-MAPS OF THE LADDER SYSTEM SPACE

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Abstract. We give a partial characterization of the continuous self-maps of the ladder system space $K_{\mathcal{S}}$. Our results show that $K_{\mathcal{S}}$ is highly nonrigid. We also discuss reasonable notions of "few operators" for spaces $C(K)$ with scattered K and we show that $C(K_{\mathcal{S}})$ does not have few operators for such notions.

1. Introduction

The ladder system space $K_{\mathcal{S}}$ is a standard example of a compact scattered space of finite (Cantor–Bendixson) height ([1, pg. 164]). It was used by R. Pol [7] to obtain the first example of a weakly Lindelöf Banach space $C(K)$ that is not WCG. The space $K_{\mathcal{S}}$ is the one-point compactification $\omega_1 \cup \{\infty\}$ of the first uncountable ordinal ω_1 endowed with the ladder system topology $\tau_{\mathcal{S}}$ (see Section 2). A fairly uninteresting type of continuous self-map ϕ of $K_{\mathcal{S}}$ consists of those that are "almost constant" in the sense that they have countable range (and thus yield "small" composition operators $\mathcal{C}_\phi : f \mapsto f \circ \phi$ on $C(K_{\mathcal{S}})$ of separable range). Those include the continuous maps that do not fix the point at infinity ∞. One might wonder whether $K_{\mathcal{S}}$ is "almost rigid" in the sense that the continuous self-maps of $K_{\mathcal{S}}$ are either "almost constant" or are close to the identity, i.e., have lots of fixed points. We answer this question in the negative, providing a simple example of a continuous self-map of $K_{\mathcal{S}}$ having uncountable range and whose only fixed point is ∞ (Example 2). We will show that the existence of such a map already implies that $C(K_{\mathcal{S}})$ does not have few operators for some natural notions of "few operators" (Section 3). Even though maps of uncountable range may fix only the point at infinity, we prove that if $\phi : K_{\mathcal{S}} \to K_{\mathcal{S}}$ is a continuous map fixing ∞ then there must exist a club $F \subset \omega_1$ such that for all $\alpha \in F$ either $\phi(\alpha) = \alpha$, or $\phi(\alpha) = \infty$ (Proposition 1).

Another natural question is whether a continuous self-map of $K_{\mathcal{S}}$ can move uncountably many limit ordinals while not mapping them to the infinity point. We answer the latter affirmatively, showing that given a club subset F of the limit ordinals $L(\omega_1)$ then on the nonstationary set $L(\omega_1) \setminus F$ one is reasonably free to choose the value of a continuous map (Theorem 3).

Date: January 23rd, 2013.

2010 *Mathematics Subject Classification.* 54G12, 46E15.

The first author is sponsored by FAPESP (Processo n. 2010/17853-8).

2. Main results

We denote by $L(\omega_1)$ the subset of ω_1 consisting of limit ordinals and we set $S(\omega_1) = \omega_1 \setminus L(\omega_1)$. By a *ladder system* $\mathcal{S}$ on ω_1 we mean a family $\mathcal{S} = (s_\alpha)_{\alpha \in L(\omega_1)}$ where, for each limit ordinal α, we have $s_\alpha = \{s_\alpha^n : n \in \omega\}$ and $(s_\alpha^n)_{n\in\omega}$ is a strictly increasing sequence in $S(\omega_1)$ order-converging to α. The *ladder system topology* $\tau_{\mathcal{S}}$ on ω_1 is the one for which the elements of $S(\omega_1)$ are isolated and the fundamental neighborhoods of a limit ordinal α are unions of $\{\alpha\}$ with sets cofinite in s_α. Then $S(\omega_1)$ is a discrete open subset of $\omega_1^{\mathcal{S}} = (\omega_1, \tau_{\mathcal{S}})$ and $L(\omega_1)$ is a discrete closed subset of $\omega_1^{\mathcal{S}}$. The topology $\tau_{\mathcal{S}}$ is finer than the order topology. The relatively compact subsets of $\omega_1^{\mathcal{S}}$ are those which are almost contained in a finite union of ladders s_α (with "almost" meaning "except for a finite subset of ω_1"). The sequence $(s_\alpha^n)_{n\in\omega}$ converges to α in $\omega_1^{\mathcal{S}}$ and a map f defined in $\omega_1^{\mathcal{S}}$ (taking values in an arbitrary space) is continuous if and only if $\big(f(s_\alpha^n)\big)_{n\in\omega}$ converges to $f(\alpha)$, for all $\alpha \in L(\omega_1)$. The space $\omega_1^{\mathcal{S}}$ is locally compact Hausdorff of height 2 and we denote by $K_{\mathcal{S}} = \omega_1^{\mathcal{S}} \cup \{\infty\}$ its one-point compactification (which is compact Hausdorff of height 3). Since every point of $\omega_1^{\mathcal{S}}$ has a countable neighborhood and compact subsets of $\omega_1^{\mathcal{S}}$ are countable, a continuous map $\phi : K_{\mathcal{S}} \to K_{\mathcal{S}}$ with $\phi(\infty) \neq \infty$ has countable range. We are thus interested in maps ϕ fixing the point ∞.

In what follows, for a subset of ω_1, *club* is the abbreviation for *closed unbounded.*

Proposition 1. *Let $\phi : K_{\mathcal{S}} \to K_{\mathcal{S}}$ be a continuous map with $\phi(\infty) = \infty$. Then there exists a club $F \subset \omega_1$ such that for all $\alpha \in F$ either $\phi(\alpha) = \alpha$, or $\phi(\alpha) = \infty$.*

Proof. For $\alpha \in \omega_1$, the level set $\phi^{-1}(\alpha)$ is a compact subset of $\omega_1^{\mathcal{S}}$ and therefore countable. Thus the Pressing Down Lemma ([6, Lemma II.6.15]) implies that the set $\{\alpha \in \omega_1 : \phi(\alpha) < \alpha\}$ is nonstationary. Let F_1 be a club disjoint from the latter. Consider the map $\psi : \omega_1 \to \omega_1$ given by $\psi(\alpha) = \phi(\alpha)$ if $\phi(\alpha) \neq \infty$ and $\psi(\alpha) = 0$ otherwise. By standard arguments, the set F_2 of ordinals α invariant by ψ (i.e., $\beta < \alpha$ implies $\psi(\beta) < \alpha$) is club. For $\alpha \in F_2 \cap L(\omega_1)$, we have that $\phi[s_\alpha] = \{\phi(s_\alpha^n) : n \in \omega\}$ is contained in the closed set $[0, \alpha] \cup \{\infty\}$ and therefore also $\phi(\alpha)$ is in $[0, \alpha] \cup \{\infty\}$. The conclusion is obtained by taking $F = F_1 \cap F_2 \cap L(\omega_1)$. $\square$

Proposition 1 leads to the question, whether, for ϕ having uncountable range, the set of fixed points of ϕ contains a club. The following example shows that this is not the case.

Example 2. Let $\phi : K_{\mathcal{S}} \to K_{\mathcal{S}}$ be a map whose restriction to $S(\omega_1)$ is an injection into $L(\omega_1)$ and that maps every element of $L(\omega_1) \cup \{\infty\}$ to ∞. For limit $\alpha \in \omega_1$, the sequence $\big(\phi(s_\alpha^n)\big)_{n\in\omega}$ is injective and contained in $L(\omega_1)$; therefore it converges to $\infty = \phi(\alpha)$. This proves the continuity of ϕ at the points of ω_1. It is easy to see that a map $\psi : K_{\mathcal{S}} \to K_{\mathcal{S}}$ with $\psi(\infty) = \infty$

is continuous at ∞ if and only if $\psi^{-1}(\alpha)$ is relatively compact in $\omega_1^{\mathcal{S}}$ for all $\alpha \in \omega_1$ and also $\psi^{-1}[s_\alpha]$ is relatively compact in $\omega_1^{\mathcal{S}}$ for all $\alpha \in L(\omega_1)$. It follows from this criterion that ϕ is continuous at ∞.

Proposition 1 says that continuous self-maps of $K_{\mathcal{S}}$ (fixing ∞) are highly constrained in a club subset F: namely every point of F is either fixed or mapped to ∞. We now show that on the nonstationary set $L(\omega_1) \setminus F$ a continuous self-map of $K_{\mathcal{S}}$ can be chosen with a lot of freedom.

Theorem 3. *Let $\xi : A \to L(\omega_1)$ be an injective map defined in a nonstationary subset A of $L(\omega_1)$. Then ξ extends to a continuous self-map ϕ of $K_{\mathcal{S}}$ that maps ∞ and every element of $L(\omega_1) \setminus A$ to ∞.*

Proof. We claim that the ladders s_α, $\alpha \in A$, can be made disjoint by removing a finite number of elements from each of them. Obviously, this will not change $\tau_{\mathcal{S}}$. Since the ladders are already almost disjoint, the claim is trivial if A is countable. In the rest of the proof of the claim, we assume that A is uncountable. Let $F \subset \omega_1$ be a club disjoint from A and, for $\alpha \in A$, let $\lambda(\alpha)$ be the largest element of $F \cup \{0\}$ below α. Then $\lambda : A \to \omega_1$ is regressive (i.e., $\lambda(\alpha) < \alpha$, for all $\alpha \in A$) and cofinal (i.e., for all $\alpha \in \omega_1$, there exists $\beta_0 \in A$, such that, for all $\beta \in A$, $\beta \geq \beta_0$ implies $\lambda(\beta) \geq \alpha$). For $\alpha \in A$, remove the finite number of elements of s_α strictly below $\lambda(\alpha)$; now s_α denotes the set thus obtained. Consider the binary relation R on A defined by $(\alpha, \beta) \in R \Leftrightarrow s_\alpha \cap s_\beta \neq \emptyset$. The fact that λ is cofinal implies that, for each $\alpha \in A$, only a countable number of elements of A are R-related to α. Hence the equivalence classes defined by the equivalence relation spanned by R (i.e., the smallest equivalence relation containing R) are also countable. The proof of the claim is now obtained by using the fact that a countable number of almost disjoint sets can be made disjoint by removing a finite number of elements from each of them.

The ladders s_α, $\alpha \in A$, are now assumed to be disjoint. Define ϕ by mapping α and each element of s_α to $\xi(\alpha)$, for $\alpha \in A$; map everything else to ∞. The continuity of ϕ at the points of $A \cup S(\omega_1)$ is clear. For $\alpha \in L(\omega_1) \setminus A$, the sequence $\big(\phi(s_\alpha^n)\big)_{n \in \omega}$ is contained in $L(\omega_1) \cup \{\infty\}$ and has no constant subsequence contained in $L(\omega_1)$. Therefore, it converges to $\infty = \phi(\alpha)$. Finally, the continuity of ϕ at ∞ follows from the criterion explained in Example 2. $\square$

In the statement of Theorem 3 the assumptions on ξ can be weakened: namely, it suffices to assume that $\xi : A \to L(\omega_1) \cup \{\infty\}$ be a map with $\xi^{-1}(\alpha)$ finite, for all $\alpha \in L(\omega_1)$. Note that this weaker assumption on ξ is necessary: if $\alpha \in L(\omega_1)$ then $\phi^{-1}(\alpha)$ is a compact subset of $\omega_1^{\mathcal{S}}$ and hence $\phi^{-1}(\alpha) \cap L(\omega_1)$ must be finite.

3. Operators on $C(K_{\mathcal{S}})$

We now discuss the bounded operators on the Banach space $C(K_{\mathcal{S}})$ of continuous real-valued maps on $K_{\mathcal{S}}$. Given a compact Hausdorff space K

and $g \in C(K)$, we denote by M_g the multiplication operator $f \mapsto fg$ on $C(K)$. If g belongs to the space $\mathfrak{B}(K)$ of real-valued bounded Borel functions on K, we denote by M_g^* the operator on $C(K)^*$ defined by $\mu \mapsto \int g \, \mathrm{d}\mu$. (When g is continuous, M_g^* is indeed the adjoint of M_g.) In [4], an operator T on $C(K)$ is called a *weak multiplication* if $T + M_g$ is weakly compact for some $g \in C(K)$ and it is called a *weak multiplier* if $T^* + M_g^*$ is weakly compact, for some $g \in \mathfrak{B}(K)$. By Gantmacher's Theorem ([2, Theorem VI.4.8]), the adjoint of a weakly compact operator is weakly compact and therefore every weak multiplication is a weak multiplier. In [4], the space $C(K)$ is said to have *few operators* if every operator is a weak multiplier. If K is infinite and scattered then $C(K)$ cannot have few operators in this sense: namely, in this case $C(K)$ has a complemented copy of c_0 ([3, Theorem 14.26]) and thus is isomorphic to its closed hyperplanes. But this is impossible if $C(K)$ has few operators ([4, Theorem 3.2]). However, in [5], it is presented (under $\clubsuit$) an example of a nonmetrizable scattered compact space K (of infinite height) for which every operator on $C(K)$ is a constant multiple of the identity plus an operator of separable range. Having this in mind, for a given compact Hausdorff space K, one can consider the question of whether every operator T on $C(K)$ satisfies one of the following conditions:

(a) $T + M_g$ has separable range, for some $g \in C(K)$;
(b) $T^* + M_g^*$ has separable range, for some $g \in \mathfrak{B}(K)$;
(c) the restriction of $(T^* + M_g^*)^*$ to $C(K)$ has separable range, for some $g \in \mathfrak{B}(K)$.

Condition (a) is a natural modification of the notion of weak multiplication and (b), (c) are natural candidates for modifications of the notion of weak multiplier. Note that (a) implies (c). However, (a) does not imply (b), as shown in Example 5 below. The following lemma gives a simple sufficient condition for a composition operator $\mathcal{C}_\phi$ not to satisfy (b).

Lemma 4. *Let K be a compact Hausdorff space and $\phi : K \to K$ be a continuous map. If there exists an uncountable subset A of K such that $\phi|_A$ is injective and $\phi(A) \cap A = \emptyset$ then $T = \mathcal{C}_\phi$ does not satisfy condition (b).*

Proof. Simply notice that if $g \in \mathfrak{B}(K)$ then $T^* + M_g^*$ maps $\{\delta_x : x \in A\}$ to an uncountable discrete set, where δ_x denotes the delta-measure supported at x. $\square$

Example 5. If $K = S^1$ is the unit circle and $\phi : K \ni x \mapsto -x \in K$ is the antipodal map then $T = \mathcal{C}_\phi$ clearly satisfies (a), because $C(K)$ is separable. However, using Lemma 4 with A equal to an open half circle, we obtain that T does not satisfy (b).

We now prove the main result of this section.

Theorem 6. *If $K = K_{\mathcal{S}}$ then there exists an operator T on $C(K)$ that does not satisfy either (b) or (c) (and hence does not satisfy (a)). More specifically, if ϕ is defined as in Example 2, then $T = \mathcal{C}_\phi$ does not satisfy either (b) or (c).*

Proof. The fact that T does not satisfy (b) follows by using Lemma 4 with $A = S(\omega_1)$. Let us prove that T does not satisfy (c). Note first that, for $g \in \mathfrak{B}(K_\mathcal{S})$, the restriction of $(T^* + M_g^*)^*$ to $C(K_\mathcal{S})$ is identified with $T + M_g : C(K_\mathcal{S}) \to \mathfrak{B}(K_\mathcal{S})$, where $\mathfrak{B}(K_\mathcal{S})$ is identified with a subspace of $C(K_\mathcal{S})^{**}$ in the natural way. Assuming that $\mathcal{C}_\phi + M_g$ has separable range, we prove first that g must vanish outside a countable set. If there were uncountably many $\alpha \in S(\omega_1)$ with $g(\alpha) \neq 0$, there would exist $\varepsilon > 0$ and an uncountable subset A of $S(\omega_1)$ with $|g(\alpha)| \geq \varepsilon$, for all $\alpha \in A$. Then $\mathcal{C}_\phi + M_g$ would map $\{\chi_{\{\alpha\}} : \alpha \in A\}$ to an uncountable discrete set (where χ denotes characteristic function). Similarly, if there were uncountably many α in $L(\omega_1)$ with $g(\alpha) \neq 0$, there would exist $\varepsilon > 0$ and an uncountable subset A of $L(\omega_1)$ with $|g(\alpha)| \geq \varepsilon$, for all $\alpha \in A$. Then $\mathcal{C}_\phi + M_g$ would map the set $\{\chi_{s_\alpha \cup \{\alpha\}} : \alpha \in A\}$ to an uncountable discrete set. We have thus proven that the support of g, $\operatorname{supp} g$, must be countable. This implies that M_g has separable range, since it factorizes through the restriction map $C(K_\mathcal{S}) \to C(\operatorname{supp} g)$ and $C(\operatorname{supp} g)$ is separable. Finally, the fact that M_g has separable range implies that $\mathcal{C}_\phi$ has separable range as well. This is a contradiction, because $\mathcal{C}_\phi\big[C(K_\mathcal{S})\big] \equiv C\big(\phi[K_\mathcal{S}]\big)$ and $\phi[K_\mathcal{S}]$ contains an uncountable discrete set. ☐

Acknowledgments. The authors wish to thank the anonymous referee for the valuable suggestions.

References

[1] A. V. Arkhangel'skii, *Topological Function Spaces*, Math. and its Appls. (Soviet Series) **78**, Kluwer Academic Publishers, (Dordrecht, 1992).

[2] N. Dunford and J. T. Schwartz, *Linear Operators, part I: General Theory*, Pure and Applied Math. **7**, Interscience Publishers, (New York, 1958).

[3] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*, CMS Books in Math., Springer, (New York, 2011).

[4] P. Koszmider, A survey on Banach spaces $C(K)$ with few operators, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, **104** (2010), no. 2, 309—326.

[5] P. Koszmider and P. Zieliński, Complementation and decompositions in some weakly Lindelöf Banach spaces, *J. Math. Anal. Appl.*, **376** (2011), no. 1, 329—341.

[6] K. Kunen, *Set Theory: An Introduction to Independence Proofs*, Studies in Logic and the Found. of Math. **102**, North-Holland Publishing Company, (Amsterdam, 1980).

[7] R. Pol, A function space $C(X)$ which is weakly Lindelöf but not weakly compactly generated, *Studia Math.*, **64** (1979), no. 3, 279—285.

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