Additional set-theoretic assumptions and twisted sums of Banach spaces

Claudia Correa
Universidade Federal do ABC – Brazil
This project was partially supported by FAPESP grants 2014/00848-2 and 2018/09797-2

1st Workshop "Studies in Mathematical Logic",
IME-USP, 05-06 december 2019

Birth of the Problem

Definition

Let X and Y be Banach spaces. A **twisted sum** *of Y and X is a short exact sequence of the form:*

$$0 \longrightarrow Y \xrightarrow{T} Z \xrightarrow{S} X \longrightarrow 0,$$

where Z is a Banach space and the maps T and S are linear and bounded.

Definition

Let X and Y be Banach spaces. A **twisted sum** *of Y and X is a short exact sequence of the form:*

$$0 \longrightarrow Y \xrightarrow{T} Z \xrightarrow{S} X \longrightarrow 0,$$

where Z is a Banach space and the maps T and S are linear and bounded.

Remark

Note that since $T[Y] = KerS$, it follows from the Open Mapping Theorem that Y is isomorphic to ImT and the quotient $Z/T[Y]$ is isomorphic to X, through $\overline{S} : Z/T[Y] \to X$.

Example

If X and Y are Banach spaces and the direct sum $Y \bigoplus X$ is endowed with some product norm, then:

$$0 \longrightarrow Y \xrightarrow{i_1} Y \bigoplus X \xrightarrow{\pi_2} X \longrightarrow 0$$

is a twisted sum of Y and X, where i_1 is the canonical embedding and π_2 is the second projection.

Example

If X and Y are Banach spaces and the direct sum $Y \bigoplus X$ is endowed with some product norm, then:

$$0 \longrightarrow Y \xrightarrow{i_1} Y \bigoplus X \xrightarrow{\pi_2} X \longrightarrow 0$$

is a twisted sum of Y and X, where i_1 is the canonical embedding and π_2 is the second projection.

Definition

A twisted sum:

$$0 \longrightarrow Y \xrightarrow{T} Z \xrightarrow{S} X \longrightarrow 0$$

of Y and X is called **trivial** *if $T[Y]$ is complemented in Z.*

Question

Are there nontrivial twisted sums of Banach spaces?

Question

Are there nontrivial twisted sums of Banach spaces?

Answer: Yes.

Question

Are there nontrivial twisted sums of Banach spaces?

Answer: Yes.

Theorem (Phillips–1940)

The sequence space c_0 is not a complemented subspace of ℓ_∞.

Question

Are there nontrivial twisted sums of Banach spaces?

Answer: Yes.

Theorem (Phillips–1940)

The sequence space c_0 is not a complemented subspace of ℓ_∞.

Corollary

The twisted sum:

$$0 \longrightarrow c_0 \xrightarrow{inc} \ell_\infty \xrightarrow{q} \ell_\infty / c_0 \longrightarrow 0$$

is not trivial, where inc denotes the inclusion map and q denotes the quotient map.

Theorem (Sobczyk–1941)

Every isomorphic copy of c_0 *inside a separable Banach space is complemented.*

Theorem (Sobczyk–1941)

Every isomorphic copy of c_0 inside a separable Banach space is complemented.

Corollary

If X is a separable Banach space, then every twisted sum of c_0 and X is trivial.

Proof. Let Z be a Banach space such that:

$$0 \longrightarrow c_0 \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is an exact sequence. In this case Z is also separable and therefore Sobczyk's Theorem ensures that this twisted sum is trivial.

Question

Let X be a Banach space. If every twisted sum of c_0 and X is trivial, then X must be separable?

Question

Let X be a Banach space. If every twisted sum of c_0 and X is trivial, then X must be separable?

Answer: No.

Question

Let X be a Banach space. If every twisted sum of c_0 and X is trivial, then X must be separable?

Answer: No.

Example

If I is an uncountable set, then the Banach space $\ell_1(I)$ is not separable but it is projective. Therefore, every twisted sum of c_0 and $\ell_1(I)$ is trivial.

Question

Let X be a Banach space. If every twisted sum of c_0 and X is trivial, then X must be separable?

Answer: No.

Example

If I is an uncountable set, then the Banach space $\ell_1(I)$ is not separable but it is projective. Therefore, every twisted sum of c_0 and $\ell_1(I)$ is trivial.

Definition

Given a compact Hausdorff space K, we denote by $C(K)$ the Banach space of continuous real-valued functions defined on K, endowed with the supremum norm.

Question

What does Sobczyk's Theorem tell us about the $C(K)$ world?

Question

What does Sobczyk's Theorem tell us about the $C(K)$ world?

Proposition

Let K be a compact Hausdorff space. The Banach space $C(K)$ is separable if and only if K is metrizable.

Question

What does Sobczyk's Theorem tell us about the $C(K)$ world?

Proposition

Let K be a compact Hausdorff space. The Banach space $C(K)$ is separable if and only if K is metrizable.

Corollary (of Sobczyk's Theorem)

If K is a metrizable compact space, then every twisted sum of c_0 and $C(K)$ is trivial.

Problem (The Golden Problem)

Is there a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial?

Problem (The Golden Problem)

Is there a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial?

This problem was proposed in 2003 by F. Cabelo, J. Castillo, N. Kalton and D. Yost and it has been completely solved only this year.

It is reasonable to believe that if we want to find a nonmetrizable compact space that shares a property with the metrizable ones, then we should start investigating nonmetrizable compact spaces that are similar to the metrizable ones.

It is reasonable to believe that if we want to find a nonmetrizable compact space that shares a property with the metrizable ones, then we should start investigating nonmetrizable compact spaces that are similar to the metrizable ones.

Proposition

A compact space K is metrizable if and only if K is homeomorphic to a $\|.\|$-compact subset of the Banach space c_0.

Idea of the proof. If K is a metrizable compact space, then $C(K)$ is separable. Let $\{f_n : n \geq 1\}$ be a dense subset of the closed unit ball of $C(K)$. The following map is an homeomorphic embedding $\varphi : K \to c_0$ defined as $\varphi(p) = \left(\frac{f_n(p)}{n}\right)_{n\geq 1}$, for every $p \in K$.

Definition

We say that a compact Hausdorff space is an **Eberlein compactum** *if there exists a set I such that K is homeomorphic to a weakly compact subset of $c_0(I)$, endowed with the weak topology.*

Definition

We say that a compact Hausdorff space is an **Eberlein compactum** *if there exists a set I such that K is homeomorphic to a weakly compact subset of $c_0(I)$, endowed with the weak topology.*

Example

- *Every metrizable compact space is an Eberlein compactum.*

Definition

We say that a compact Hausdorff space is an **Eberlein compactum** *if there exists a set I such that K is homeomorphic to a weakly compact subset of $c_0(I)$, endowed with the weak topology.*

Example

- *Every metrizable compact space is an Eberlein compactum.*
- *The one-point compactification of an uncountable discrete space is a nonmetrizable Eberlein compactum.*

Definition

We say that a compact Hausdorff space is an **Eberlein compactum** *if there exists a set I such that K is homeomorphic to a weakly compact subset of $c_0(I)$, endowed with the weak topology.*

Example

- *Every metrizable compact space is an Eberlein compactum.*
- *The one-point compactification of an uncountable discrete space is a nonmetrizable Eberlein compactum.*

Remark

Eberlein compacta share many properties with compact metrizable spaces. For instance: Eberlein compacta are Fréchet-Urysohn.

Theorem (Cabello, Castillo, Kalton and Yost, Trans. Amer. Math. Soc.)

If K is a nonmetrizable Eberlein compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Theorem (Cabello, Castillo, Kalton and Yost, Trans. Amer. Math. Soc.)

If K is a nonmetrizable Eberlein compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Remark

Recall that a compact space is metrizable if and only if it is homeomorphic to a subset of $\mathbb{R}^{\omega}$, endowed with the product topology.

Theorem (Cabello, Castillo, Kalton and Yost, Trans. Amer. Math. Soc.)

If K is a nonmetrizable Eberlein compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Remark

Recall that a compact space is metrizable if and only if it is homeomorphic to a subset of $\mathbb{R}^\omega$, endowed with the product topology.

Definition

Given a set I, we denote by $\Sigma(I)$ the subset of $\mathbb{R}^I$ formed by the functions with countable support.

Definition

We say that a compact Hausdorff space is a **Corson compactum** *if there exists a set I such that K is homeomorphic to a subset of $\Sigma(I)$, endowed with the product topology.*

Definition

We say that a compact Hausdorff space is a **Corson compactum** *if there exists a set I such that K is homeomorphic to a subset of $\Sigma(I)$, endowed with the product topology.*

Example

- *Every metrizable compact space is Corson.*

Definition

We say that a compact Hausdorff space is a **Corson compactum** *if there exists a set I such that K is homeomorphic to a subset of $\Sigma(I)$, endowed with the product topology.*

Example

- *Every metrizable compact space is Corson.*
- *Every Eberlein compact space is Corson*

Definition

We say that a compact Hausdorff space is a **Corson compactum** *if there exists a set I such that K is homeomorphic to a subset of $\Sigma(I)$, endowed with the product topology.*

Example

- *Every metrizable compact space is Corson.*
- *Every Eberlein compact space is Corson*
- *There exist Corson compact spaces that are not Eberlein.*

Definition

We say that a compact Hausdorff space is a **Corson compactum** *if there exists a set I such that K is homeomorphic to a subset of $\Sigma(I)$, endowed with the product topology.*

Example

- *Every metrizable compact space is Corson.*
- *Every Eberlein compact space is Corson*
- *There exist Corson compact spaces that are not Eberlein.*

Remark

The class of Eberlein compacta is topologically much closer to the class of metrizable compacta than the class of Corson compacta.

Question

Let K be a nonmetrizable Corson compactum. Is there a nontrivial twisted sum of c_0 and $C(K)$?

Question

Let K be a nonmetrizable Corson compactum. Is there a nontrivial twisted sum of c_0 and $C(K)$?

In the same paper where it was shown that if K is a nonmetrizable Eberlein compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$, the authors claimed that with similar arguments one could prove that if K is a nonmetrizable Corson compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Question

Let K be a nonmetrizable Corson compactum. Is there a nontrivial twisted sum of c_0 and $C(K)$?

In the same paper where it was shown that if K is a nonmetrizable Eberlein compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$, the authors claimed that with similar arguments one could prove that if K is a nonmetrizable Corson compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

It turns out that similar arguments do not work and that the situation is much more complicated.

While trying to adapt the proof for Corson compacta I realized that maybe additional set-theoretic assumptions would be needed. Because even though properties about Eberlein compacta usually are established in ZFC, many properties about Corson compacta are independent from ZFC.

While trying to adapt the proof for Corson compacta I realized that maybe additional set-theoretic assumptions would be needed. Because even though properties about Eberlein compacta usually are established in ZFC, many properties about Corson compacta are independent from ZFC.

Example

- *Every Eberlein compactum with the countable chain condition (ccc) is separable.*

While trying to adapt the proof for Corson compacta I realized that maybe additional set-theoretic assumptions would be needed. Because even though properties about Eberlein compacta usually are established in *ZFC*, many properties about Corson compacta are independent from *ZFC*.

Example

- *Every Eberlein compactum with the countable chain condition (ccc) is separable.*
- *Assuma $MA + \neg CH$. Every Corson compactum with ccc is separable.*

While trying to adapt the proof for Corson compacta I realized that maybe additional set-theoretic assumptions would be needed. Because even though properties about Eberlein compacta usually are established in ZFC, many properties about Corson compacta are independent from ZFC.

Example

- *Every Eberlein compactum with the countable chain condition (ccc) is separable.*
- *Assuma $MA + \neg CH$. Every Corson compactum with ccc is separable.*
- *Assume CH. There exists a nonseparable Corson compactum with ccc.*

Theorem (Claudia and Dani, JFA)

Assume MA. If K is a nonmetrizable Corson compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Theorem (Claudia and Dani, JFA)

Assume MA. If K is a nonmetrizable Corson compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Open Problem

Does it hold in ZFC?

Theorem (Claudia and Dani, JFA)

Assume MA. If K is a nonmetrizable Corson compactum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Open Problem

Does it hold in ZFC?

Theorem (Claudia and Dani, JFA)

Assume CH. There exists a nontrivial twisted sum of c_0 and $C(2^{\omega_1})$.

Great Surprise

Theorem (Marciszewski and Plebanek, JFA)

Assume $MA + \neg CH$. Every twisted sum of c_0 and $C(2^\kappa)$ is trivial, for $\omega_1 \leq \kappa < \mathfrak{c}$.

Great Surprise

Theorem (Marciszewski and Plebanek, JFA)

Assume $MA + \neg CH$. Every twisted sum of c_0 and $C(2^\kappa)$ is trivial, for $\omega_1 \leq \kappa < \mathfrak{c}$.

Corollary (Partial answer to the Golden Problem)

The existence of a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial is relatively consistent with ZFC.

Definition

We say that a topological space $\mathcal{X}$ is scattered if every nonempty subspace of $\mathcal{X}$ has an isolated point.

Scattered spaces

Definition

We say that a topological space $\mathcal{X}$ is scattered if every nonempty subspace of $\mathcal{X}$ has an isolated point.

Example

If Γ is a discrete topological space, then its one-point compactification $\Gamma \cup \{\infty\}$ is scattered.

Definition

Let $\mathcal{X}$ be a topological space. We define by recursion on α a decreasing family of closed subsets of $\mathcal{X}$:

- $\mathcal{X}^{(0)} = \mathcal{X}$;
- *For every ordinal α, $\mathcal{X}^{(\alpha+1)} = \mathcal{X}^{(\alpha)} \setminus Is(\mathcal{X}^{(\alpha)})$, where $Is(\mathcal{X}^{(\alpha)})$ denotes the set of isolated points of $X^{(\alpha)}$;*
- *For every limit ordinal α, $\mathcal{X}^{(\alpha)} = \bigcap_{\beta\in\alpha} \mathcal{X}^{(\beta)}$.*

The space $\mathcal{X}^{(\alpha)}$ is called the α^{th} **Cantor–Bendixson derivative** *of $\mathcal{X}$*

Definition

Let $\mathcal{X}$ be a topological space. We define by recursion on α a decreasing family of closed subsets of $\mathcal{X}$:

- $\mathcal{X}^{(0)} = \mathcal{X}$;
- *For every ordinal α, $\mathcal{X}^{(\alpha+1)} = \mathcal{X}^{(\alpha)} \setminus Is(\mathcal{X}^{(\alpha)})$, where $Is(\mathcal{X}^{(\alpha)})$ denotes the set of isolated points of $X^{(\alpha)}$;*
- *For every limit ordinal α, $\mathcal{X}^{(\alpha)} = \bigcap_{\beta \in \alpha} \mathcal{X}^{(\beta)}$.*

The space $\mathcal{X}^{(\alpha)}$ is called the α^{th} **Cantor–Bendixson derivative** *of $\mathcal{X}$*

Proposition

A topological space $\mathcal{X}$ is scattered if and only if there exists an ordinal α such that $\mathcal{X}^{(\alpha)} = \emptyset$.

Definition

If $\mathcal{X}$ is scattered, then the **height** *of $\mathcal{X}$ is defined as the least ordinal α such that $\mathcal{X}^{(\alpha)} = \emptyset$. If the height of $\mathcal{X}$ is a natural number, then we say that $\mathcal{X}$ has* **finite height***.*

Definition

If $\mathcal{X}$ is scattered, then the **height** *of $\mathcal{X}$ is defined as the least ordinal α such that $\mathcal{X}^{(\alpha)} = \emptyset$. If the height of $\mathcal{X}$ is a natural number, then we say that $\mathcal{X}$ has* **finite height**.

Theorem (Castillo, Top. Appl.)

Assume CH. If K is a nonmetrizable finite height compact Hausdorff space, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Definition

If $\mathcal{X}$ is scattered, then the **height** *of $\mathcal{X}$ is defined as the least ordinal α such that $\mathcal{X}^{(\alpha)} = \emptyset$. If the height of $\mathcal{X}$ is a natural number, then we say that $\mathcal{X}$ has* **finite height**.

Theorem (Castillo, Top. Appl.)

Assume CH. If K is a nonmetrizable finite height compact Hausdorff space, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Theorem (Marciszewski and Plebanek, JFA)

Assume MA+$\neg$ CH. If K is a separable scattered compact Hausdorff space with height 3 and weight smaller than the continuum, then every twisted sum of c_0 and $C(K)$ is trivial.

Theorem (Claudia and Dani, Fund. Math.)

Assume MA+$\neg$ CH. If K is a separable finite height compact Hausdorff space and weight smaller than the continuum, then every twisted sum of c_0 and $C(K)$ is trivial.

Theorem (Claudia and Dani, Fund. Math.)

Assume MA+$\neg$ CH. If K is a separable finite height compact Hausdorff space and weight smaller than the continuum, then every twisted sum of c_0 and $C(K)$ is trivial.

Corollary

The existence of nontrivial twisted sums of c_0 and $C(K)$, for every separable finite height compact Hausdorff space with weight smaller than the continuum, is independent from ZFC.

Theorem (Claudia and Dani, Fund. Math.)

Assume MA+$\neg$ CH. If K is a separable finite height compact Hausdorff space and weight smaller than the continuum, then every twisted sum of c_0 and $C(K)$ is trivial.

Corollary

The existence of nontrivial twisted sums of c_0 and $C(K)$, for every separable finite height compact Hausdorff space with weight smaller than the continuum, is independent from ZFC.

Theorem (Claudia, Fund. Math)

Assume $MA + \neg CH$. Let K be a scattered compact Hausdorff space with weight smaller than the continuum. If K is nonseparable, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Open Problem

Does it hold in ZFC?

Open Problem

Does it hold in ZFC?

Theorem (Claudia, Fund. Math)

Assume $MA + \neg CH$. If K is a finite height compact Hausdorff space with weight greater or equal to the continuum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Open Problem

Does it hold in ZFC?

Theorem (Claudia, Fund. Math)

Assume $MA + \neg CH$. If K is a finite height compact Hausdorff space with weight greater or equal to the continuum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Problem

Does this hold in ZFC?

Open Problem

Does it hold in ZFC?

Theorem (Claudia, Fund. Math)

Assume $MA + \neg CH$. If K is a finite height compact Hausdorff space with weight greater or equal to the continuum, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Problem

Does this hold in ZFC?

Answer: Yes. It was shown by Aviles, Marciszewski and Plebanek this year.

Another great surprise

Problem

Is there in ZFC a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial?

Another great surprise

Problem

Is there in ZFC a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial?

Theorem (Aviles, Marciszewski and Plebanek)

Assume CH. If K is a nonmetrizable compact Hausdorff space, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Another great surprise

Problem

Is there in ZFC a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial?

Theorem (Aviles, Marciszewski and Plebanek)

Assume CH. If K is a nonmetrizable compact Hausdorff space, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Theorem (Answer to the Golden Problem)

The existence of a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial is independent from ZFC.


THANK YOU!

A. Avilés, W. Marciszewski, and G. Plebanek.
Twisting c_0 around nonseparable banach spaces.
preprint at https://arxiv.org/pdf/1902.07783.pdf, 2019.

J. M. Castillo.
Nonseparable $c(k)$-spaces can be twisted when k is a finite height compact.
Top. Appl., 198:107–116, 2016.

F. Cabello, J. M Castillo, N. J. Kalton, and D. T. Yost.
Twisted sums with $c(k)$ spaces.
Trans. Amer. Math. Soc., 355 (11):4523–4541, 2003.

C. Correa.
Nontrivial twisted sums for finite height spaces under martin's axiom.
To appear in Fund. Math.

C. Correa and D. V. Tausk.
Nontrivial twisted sums of c_0 and $c(k)$.
J. Funct. Anal., 270 (2):842–853, 2016.

C. Correa and D. V. Tausk.
Local extension property for finite height spaces.
Fund. Math., 245:149–165, 2019.

W. Marciszewski and G. Plebanek.
Extension operators and twisted sums of c_0 and $c(k)$ spaces.
J. Funct. Anal., 274 (5):1491–1529, 2018.