Twisted sums of c_0 and $C(K)$
Joint work with Daniel Tausk

Claudia Correa

Universidade Federal do ABC—Brazil

claudia.correa@ufabc.edu.br

5 de julho de 2018

Definition

Let X and Y be Banach spaces. A twisted sum of Y and X *is a short exact sequence of the form:*

$$0 \longrightarrow Y \xrightarrow{T} Z \xrightarrow{S} X \longrightarrow 0,$$

where Z is a Banach space and the maps T and S are linear and bounded.

Remark

Note that since $T[Y] = KerS$, it follows from the Open Mapping Theorem that Y is isomorphic to $T[Y]$ and the quotient $Z/T[Y]$ is isomorphic to X, through $\overline{S} : Z/T[Y] \to X$.

Example

If X and Y are Banach spaces and the direct sum $Y \bigoplus X$ is endowed with some product norm, then:

$$0 \longrightarrow Y \xrightarrow{i_1} Y \bigoplus X \xrightarrow{\pi_2} X \longrightarrow 0$$

is a twisted sum of Y and X, where i_1 is the canonical embedding and π_2 is the second projection.

Definition

A twisted sum:

$$0 \longrightarrow Y \xrightarrow{T} Z \xrightarrow{S} X \longrightarrow 0$$

of Banach spaces Y and X is called trivial *if $T[Y]$ is complemented in Z.*

Question

Are there nontrivial twisted sums of Banach spaces?

Answer: Yes.

Theorem (Phillips–1940)

The sequence space c_0 is not a complemented subspace of ℓ_∞.

Corollary

The twisted sum:

$$0 \longrightarrow c_0 \xrightarrow{inc} \ell_\infty \xrightarrow{q} \ell_\infty / c_0 \longrightarrow 0,$$

is not trivial, where inc denotes the inclusion map and q denotes the quotient map.

Theorem (Sobczyk–1941)

Every isomorphic copy of c_0 inside a separable Banach space is complemented.

Corollary

If X is a separable Banach space, then every twisted sum of c_0 and X is trivial.

Proof. Let Z be a Banach space such that:

$$0 \longrightarrow c_0 \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is an exact sequence. In this case Z is separable and therefore this twisted sum is trivial. □

Definition

Given a compact Hausdorff space K, we denote by $C(K)$ the Banach space of continuous real-valued functions defined on K, endowed with the supremum norm.

Proposition

Let K be a compact Hausdorff space. The Banach space $C(K)$ is separable if and only if K is metrizable.

Corollary (Corollary of Sobczyk's Theorem)

If K is a metrizable compact space, then every twisted sum of c_0 and $C(K)$ is trivial.

X separable $\Rightarrow$ every twisted sum of c_0 and X is trivial

Question

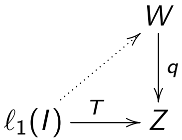
Let X be a Banach space. If every twisted sum of c_0 and X is trivial, then X must be separable?

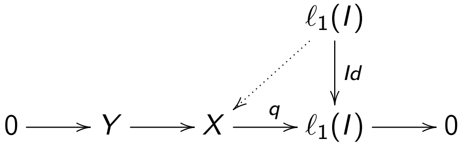
Answer: No.

Proposition

If I is an uncountable set, then the Banach space $\ell_1(I)$ is not separable and every twisted sum of c_0 and $\ell_1(I)$ is trivial.

Proof. The space $\ell_1(I)$ is a *projective* Banach space, i. e., if W and Z are Banach spaces and $q : W \longrightarrow Z$ is a quotient map, then every bounded operator $T : \ell_1(I) \longrightarrow Z$ admits a lifting:





□

K metrizable $\Rightarrow$ every twisted sum of c_0 and $C(K)$ is trivial

Open Problem (Cabelo, Castillo, Kalton and Yost–2003)

Is there a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial?

This problems remains open, but we are working on it!

Remark

If K is a compact metric space, then K is homeomorphic to a $\|.\|$-compact subset of a Banach space.

Definition

A compact space is said an Eberlein compactum *if it is homeomorphic to a weakly compact subset of a Banach space, endowed with the weak topology.*

Example

Every metrizable compact space is Eberlein and the one-point compactification of an uncountable discrete space is a nonmetrizable Eberlein compactum.

Remark

Eberlein compacta share many properties with compact metrizable spaces. For instance: If K is an Eberlein compact space, then K is a sequential space.

Theorem (Cabello, Castillo, Kalton and Yost–2003)

If K is a nonmetrizable Eberlein compact space, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

In the same paper, the authors claimed that with similar arguments one could prove that if K is a nonmetrizable Corson compact space, then there exists a nontrivial twisted sum of c_0 and $C(K)$.
It turns out that similar arguments do not work and that the situation is much more complicated.

Theorem (Amir and Lindenstrauss–1968)

A compact space K is an Eberlein compactum if and only if K is homeomorphic to a weakly compact subset of the Banach space $c_0(\Gamma)$, for some index set Γ.

Corollary

If K is an Eberlein compactum, then K is homeomorphic to a compact subspace of $c_0(\Gamma)$, endowed with the product topology.

Remark

This copy of K is contained in $\Sigma(\Gamma)$, where:

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^{\Gamma} : x \text{ has countable support}\}.$$

Definition

A compact space is called a Corson compact space *if it is homeomorphic to a subset of* $\Sigma(\Gamma)$*, endowed with the product topology, for some index set* Γ.

Remark

Every Eberlein compact space is Corson, but there are Corson compact spaces that are not Eberlein.

Theorem (Correa and Tausk, JFA–2016)

Assume MA. If K *is a nonmetrizable Corson compact space, then there exists a nontrivial twisted sum of* c_0 *and* $C(K)$.

Open Problem

Does it hold in ZFC that there exists a nontrivial twisted sum of c_0 and $C(K)$, for every nonmetrizable Corson compact space?

Definition

A Compact space K is called a Valdivia compactum *if there exists a continuous and injective map $\varphi : K \longrightarrow \mathbb{R}^{\Gamma}$ such that $\varphi^{-1}\big[\Sigma(\Gamma)\big]$ is dense in K. In this case, $\varphi^{-1}\big[\Sigma(\Gamma)\big]$ is called a dense* Σ-subset *of K.*

Example

Every Corson compact space is Valdivia. Examples of Valdivia spaces that are not Corson are given by the product spaces 2^{κ}, for any uncountable κ.

Theorem (Correa and Tausk, JFA–2016)

Assume CH. Let K be a Valdivia compact space. If K satisfies any of the following properties, then there exists a nontrivial twisted sum of c_0 and $C(K)$:

- *K has a G_δ point with no second countable neighborhoods;*
- *K has a dense Σ-subset A such that some point of $K \setminus A$ is the limit of a nontrivial sequence in K.*

Theorem (Correa and Tausk, JFA–2016)

There exists a nontrivial twisted sum of c_0 and $C(2^{\mathfrak{c}})$. Therefore, under CH, there exists a nontrivial twisted sum of c_0 and $C(2^{\omega_1})$.

Theorem (Marciszewski and Plebanek, JFA–2018)

Assume MA+$\neg$ CH. Every twisted sum of c_0 and $C(2^\kappa)$ is trivial, for $\omega_1 \leq \kappa < \mathfrak{c}$.

Corollary

It is consistent with ZFC that there is a nonmetrizable compact space K such that every twisted sum of c_0 and $C(K)$ is trivial.

Open Problem

Is there in ZFC a nonmetrizable compact space K such that every twisted sum of c_0 and $C(K)$ is trivial?

Definition

We say that a topological space $\mathcal{X}$ is scattered *if there exists an ordinal α such that its α-Cantor-Bendixson derivative $X^{(\alpha)}$ is empty. If $\mathcal{X}$ is scattered, then the least ordinal α such that $X^{(\alpha)} = \emptyset$ is called the* height *of $\mathcal{X}$. We say that $\mathcal{X}$ has* finite height *if its height is a natural number.*

Theorem (Castillo, Top. Appl.–2016)

Assume CH. If K is a nonmetrizable compact space with finite height, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Theorem (Marciszewski and Plebanek, JFA–2018)

Assume MA+$\neg$ CH. If K is a separable compact space with height 3 and weight smaller than $\mathfrak{c}$, then every twisted sum of c_0 and $C(K)$ is trivial.

Theorem (Correa and Tausk, Fund. Math.–2018)

Assume MA+$\neg$ CH. If K is a separable compact space with finite height and weight smaller than $\mathfrak{c}$, then every twisted sum of c_0 and $C(K)$ is trivial.

Corollary

The existence of nontrivial twisted sums of c_0 and $C(K)$, where K is a finite height separable compact space, is independent of ZFC.

Theorem (Marciszewski and Plebanek, JFA–2018)

Assume CH. If K is a nonseparable scattered space, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Open Problem

Does it hold in ZFC that if K is a nonseparable scattered space, then there exists a nontrivial twisted sum of c_0 and $C(K)$?

Proposition (Correa–work in preparation)

Assume MA+$\neg$ CH. If K is a nonseparable scattered space of weight smaller than $\mathfrak{c}$, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Open Problem

Assuming MA+¬CH, is there a nontrivial twisted sum of c_0 and $C(K)$, for every nonseparable scattered compact space K?

Conjecture (My personal conjecture)

If K is a compact space with weight greater or equal to $\mathfrak{c}$, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

J. M Castillo.
Nonseparable $c(k)$-spaces can be twisted when k is a finite height compact.
Topology Appl., 198:107–116, 2016.

F. Cabello, J. M Castillo, N. J. Kalton, and D. T. Yost.
Twisted sums with $c(k)$ spaces.
Trans. Amer. Math. Soc., 355 (11):4523–4541, 2003.

C. Correa and D. V. Tausk.
Nontrivial twisted sums of c_0 and $c(k)$.
J. Func. Anal., 270:842–853, 2016.

C. Correa and D. V. Tausk.
Local extension property for finite height spaces.
To appear Fund. Math., 2018.

W. Marciszewski and G. Plebanek.
Extension operators and twisted sums of c_0 and $c(k)$ spaces.
J. Funct. Anal., 274 (5):1491–1529, 2018.