

COMPACT LINES AND THE SOBCZYK PROPERTY

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ABSTRACT. We show that Sobczyk's Theorem holds for a new class of Banach spaces, namely spaces of continuous functions on linearly ordered compacta.

1. INTRODUCTION

The main result of this article (Corollary 2.5) states that, for every compact line K , the Banach space $C(K)$ of continuous real-valued functions on K (endowed with the supremum norm) has the *Sobczyk property*, i.e., every isomorphic copy of c_0 in $C(K)$ is complemented. By a *compact line* we mean a linearly ordered set which is compact in the order topology (see [2, 1.7.4] for the definition and [2, 3.12.3] for the characterization of linear orders yielding a compact topology). Topological properties of compact lines and structural properties of their spaces of continuous functions have recently been studied in a series of articles [1, 6, 7, 10].

In his celebrated theorem [16], Sobczyk has proven that every separable Banach space has the property now named after him. Such result follows from the fact that c_0 is *separably injective*, i.e., given a separable Banach space X and a closed subspace Y of X , every bounded operator $T : Y \rightarrow c_0$ admits a bounded extension to X . When the latter condition holds for a closed subspace Y of a Banach space X , we say that Y has the *c_0 -extension property* (briefly, c_0 EP) in X . Moreover, a Banach space X is said to have the (resp., *separable*) *c_0 -extension property* if every (resp., separable) closed subspace of X has the c_0 EP in X . The c_0 EP for Banach spaces has been studied by the authors in [1].

The main result of the present article follows from our Theorem 2.4 which states that, if K is a compact line, then Banach subspaces of $C(K)$ with separable dual have the c_0 EP in $C(K)$. The latter generalizes [1, Theorem 3.1], which implies that, for a compact line K , Banach subalgebras of $C(K)$ with separable dual have the c_0 EP in $C(K)$ (though — see Remark 2.8 — the extension constant obtained in [1, Theorem 3.1] is smaller). Note that, although separable Banach subspaces of $C(K)$ always span separable Banach

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subalgebras, even a one-dimensional subspace of $C(K)$ can span a Banach subalgebra with nonseparable dual. (By the *Banach subalgebra spanned* by a subset of $C(K)$ we mean the smallest Banach subalgebra of $C(K)$ containing that set.) Thus Theorem 2.4 is much stronger than [1, Theorem 3.1]: the latter does not imply that $C(K)$ has the Sobczyk property.

Let us briefly review some known results stating that certain classes of Banach spaces have the Sobczyk property. We remark that their proofs can usually be adapted to establish the c_0 EP or the separable c_0 EP. For instance, a simple adaptation of Veech's proof ([19]) of Sobczyk's Theorem shows that weakly compactly generated (briefly, WCG) Banach spaces have the c_0 EP (see also [1, Proposition 2.2] for details). Moltó's argument ([13, Theorem 3]) shows that, assuming a certain topological condition for the dual ball (B_{X^*}, w^*) , one obtains that the Banach space X has the separable c_0 EP. (Moltó's topological condition is satisfied, for instance, by all Corson compacta.) A Banach space X is said to satisfy the *separable complementation property* (briefly: SCP) if every separable subspace of X is contained in a separable complemented Banach subspace of X . Sobczyk's Theorem implies that all Banach spaces with the SCP have the separable c_0 EP. In [18, Lemma, pg. 494] it is shown that if K is a Valdivia compact space then $C(K)$ has the SCP. Since the separable c_0 EP is hereditary to closed subspaces, it follows also that $C(K)$ has the separable c_0 EP when K is a continuous image of a Valdivia compactum.

Recall that a compact Hausdorff space K is said to be $\aleph_0$ -*monolithic* if every separable subspace of K is metrizable. It is easy to show (see [1, Corollary 2.7]) that if K is $\aleph_0$ -monolithic then $C(K)$ has the separable c_0 EP. It turns out that if K is a compact line then $C(K)$ has the separable c_0 EP only in the trivial case when K is $\aleph_0$ -monolithic (Theorem 2.2), though $C(K)$ always has the Sobczyk property.

The *double arrow space* $DA = [0, 1] \times \{0, 1\}$ (endowed with the lexicographic order and order topology) is a separable nonmetrizable compact line and thus every compact line containing a homeomorphic copy of DA is not $\aleph_0$ -monolithic. To the best of the authors knowledge, $C(DA)$ is the first known example of a $C(K)$ space satisfying the Sobczyk property, yet not having the separable c_0 EP. We note that Patterson [14] has shown that every *isometric* copy of c_0 in $C(DA)$ is complemented, but the problem of determining whether $C(DA)$ has the Sobczyk property remained open. Patterson's argument uses only the fact that closed subsets of DA admit bounded extension operators and therefore it works for arbitrary compact lines, though it is of no use to establish complementation of arbitrary *isomorphic* copies of c_0 . In fact, in his review [5] of [14], Godefroy mentions that no examples of uncomplemented isomorphic copies of c_0 in $C(DA)$ are given in the paper. This observation led the authors to study the problem whose solution is presented in this article.

This paper is organized as follows. In Section 2 we give a characterization of the compact lines K for which $C(K)$ has the (separable) c_0 EP

(Theorem 2.2) and we prove the main theorem of the paper (Theorem 2.4). The heart of the proof of Theorem 2.4 is Lemma 2.7, whose proof takes all of Section 3.

2. THE c_0 -EXTENSION PROPERTY FOR COMPACT LINES

We start by fixing the terminology and notation for the article and by recalling a few elementary facts.

Given a compact Hausdorff space K , we identify as usual the dual space of $C(K)$ with the space $M(K)$ of finite countably-additive signed regular Borel measures on K , endowed with the total variation norm $\|\mu\| = |\mu|(K)$ ([15, Theorem 6.19]). Given a point $p \in K$ and a subset A of K , we denote by $\delta_p \in M(K)$ the probability measure with support $\{p\}$ and by χ_A the characteristic function of A .

It is easily proven using the Stone–Weierstrass Theorem (see, for instance, [9, proof of Theorem 5.4]) that all Banach subalgebras (with unity) of $C(K)$ are images $q^*C(L)$ of composition operators $q^* : C(L) \ni f \mapsto f \circ q \in C(K)$, where L is a compact Hausdorff space and $q : K \rightarrow L$ is a continuous surjection. The operator q^* is an isometric embedding and a Banach algebra homomorphism.

Bounded operators $T : C(K) \rightarrow \ell_\infty$ are always identified with bounded sequences of measures $(\mu_n)_{n \geq 1}$ in $M(K)$, where μ_n represents the n -th coordinate functional of T . In this case we will say that T is associated with $(\mu_n)_{n \geq 1}$. Note that T takes values in c_0 if and only if $(\mu_n)_{n \geq 1}$ is weak*-null.

When K is a compact line, we always denote by 0 the minimum element of K . A point $t \in K$ is said to be *right-isolated* (resp., *left-isolated*) in K if either t is the maximum (resp., minimum) element of K or t admits a successor (resp., predecessor) in K . The clopen subsets of K are finite disjoint unions of clopen intervals of K , which are of the form $[0, b]$ or $]b', b]$, with $b, b' \in K$ right-isolated.

Given a subset Q of $[0, 1]$, we denote by $\text{DA}(Q)$ the set:

$$\text{DA}(Q) = ([0, 1] \times \{0\}) \cup (Q \times \{1\}) \subset [0, 1] \times \{0, 1\}$$

endowed with the lexicographic order (and the order topology). Then $\text{DA}(Q)$ is a separable compact line and the first projection $\pi_1 : \text{DA}(Q) \rightarrow [0, 1]$ is a continuous increasing surjection. The proof of the next lemma uses a criterion for the extensibility of c_0 -valued operators that will be proven later in Section 3.

Lemma 2.1. *If $Q \subset [0, 1]$ is uncountable, then $\pi_1^*C([0, 1])$ does not have the c_0 EP in $C(\text{DA}(Q))$.*

Proof. Let $([a_n, b_n])_{n \geq 1}$ be a sequence of intervals contained in $[0, 1]$ such that $\lim_{n \rightarrow +\infty} (b_n - a_n) = 0$ and such that every $t \in [0, 1[$ belongs to infinitely many $[a_n, b_n]$. Setting $\mu_n = \delta_{a_n} - \delta_{b_n}$, then $(\mu_n)_{n \geq 1}$ is weak*-null in $M([0, 1])$ and thus we have an operator $T : C([0, 1]) \rightarrow c_0$ associated with $(\mu_n)_{n \geq 1}$. It follows from the proof of Lemma 3.2 (see Remark 3.3) that there

is no bounded operator $T' : C(\text{DA}(Q)) \rightarrow c_0$ such that $T' \circ \pi_1^* = T$. The conclusion follows. $\square$

The following result characterizes compact lines K for which $C(K)$ has the (separable) $c_0\text{EP}$.

Theorem 2.2. *Let K be a compact line. The following statements are equivalent:*

- (a) K is $\aleph_0$ -monolithic;
- (b) $C(K)$ has the $c_0\text{EP}$;
- (c) $C(K)$ has the separable $c_0\text{EP}$.

Proof. Assume (a) and let us prove (b). Let $T : X \rightarrow c_0$ be a bounded operator defined on a closed subspace X of $C(K)$. Let $S : C(K) \rightarrow \ell_\infty$ be a bounded extension of T . We will show that the quotient $C(K)/S^{-1}[c_0]$ is separable and it will follow from [1, Proposition 2.2, (a)] that $S|_{S^{-1}[c_0]}$ has a c_0 -valued bounded extension to $C(K)$. The operator S is associated with a bounded sequence $(\mu_n)_{n \geq 1}$ in $M(K)$. It follows from [6, Lemma 2.1] that $\text{supp } \mu_n$ is separable for all n and thus the closure F of $\bigcup_{n=1}^\infty \text{supp } \mu_n$ is also separable; by (a), F is metrizable. The subspace $C(K|F)$ of $C(K)$ consisting of the functions that vanish on F is clearly contained in $\text{Ker}(S) \subset S^{-1}[c_0]$. Since $C(K)/C(K|F) \equiv C(F)$ is separable, it follows that also $C(K)/S^{-1}[c_0]$ is separable.

Clearly, (b) implies (c). Now assume (c) and let us prove (a). Assuming by contradiction that K is not $\aleph_0$ -monolithic, it follows from [6, Lemma 2.5] that K contains a homeomorphic copy of $\text{DA}(Q)$, for some uncountable subset Q of $[0, 1]$. Since K is a compact line, by [10, Lemma 4.2], every closed subset of K admits a bounded extension operator and therefore $C(K)$ contains an isomorphic copy of $C(\text{DA}(Q))$. The separable $c_0\text{EP}$ is hereditary to closed subspaces and thus $C(\text{DA}(Q))$ would also have the separable $c_0\text{EP}$, contradicting Lemma 2.1. $\square$

Remark 2.3. As mentioned in the introduction, WCG Banach spaces have the $c_0\text{EP}$. Let us see that the converse does not hold. Recall that a Banach space $C(K)$ is WCG if and only if K is an Eberlein compact space ([4, Theorem 14.9]). There are examples of $\aleph_0$ -monolithic compact lines that are not Eberlein compact spaces. For instance, an ordinal segment $[0, \alpha]$ is $\aleph_0$ -monolithic; namely, it is easily proven by induction on α that the closure of a countable subset of $[0, \alpha]$ is countable. Moreover, if α is uncountable, then $[0, \alpha]$ is not a Fréchet topological space ([3, Definition 4.48]) and hence it is not an Eberlein compact space ([3, Theorem 4.50]). It follows from Theorem 2.2 that there are $C(K)$ spaces that have the $c_0\text{EP}$ but are not WCG.

Our main result follows trivially from the next theorem.

Theorem 2.4. *If K is a compact line then every Banach subspace of $C(K)$ with separable dual has the $c_0\text{EP}$ in $C(K)$.*

Corollary 2.5. *If K is a compact line then the space $C(K)$ has the Sobczyk property.* $\square$

The proof of Theorem 2.4 requires two lemmas. The first one is a simple adaptation of [6, Lemma 3.1].

Lemma 2.6. *Let K be a zero-dimensional compact line and X be a separable Banach subspace of $C(K)$. Then there exists a metrizable zero-dimensional compact line L and a continuous increasing surjection $q : K \rightarrow L$ such that X is contained in $q^*C(L)$.*

Proof. Since K is zero-dimensional, the set:

$$(1) \quad \{\chi_{[0,s]} : s \text{ right-isolated in } K\}$$

spans a dense subspace of $C(K)$. By the separability of X , there exists a countable subset E of (1) such that X is contained in the Banach space spanned by E . The Banach algebra spanned by E is of the form $q^*C(L)$, with L a compact metrizable space and $q : K \rightarrow L$ a continuous surjection. Since the elements of E are simple functions, the space L is zero-dimensional. Finally, the fact that the elements of E are monotone functions implies that $q^{-1}(t)$ is a closed interval of K , for all $t \in L$. Hence there exists a unique order on L such that q is increasing and the order-topology coincides with the topology of L . $\square$

The hard work in proving Theorem 2.4 lies within the proof of the next lemma. Such proof will be presented in Section 3.

Lemma 2.7. *Let K and L be zero-dimensional compact lines, $q : K \rightarrow L$ be a continuous increasing surjection and X be a Banach space with separable dual. Assume that L is metrizable. Given bounded operators $T : C(L) \rightarrow c_0$, $R : X \rightarrow C(L)$ and $\varepsilon, \varepsilon' > 0$, there exist bounded operators $T' : C(K) \rightarrow c_0$ and $S : C(L) \rightarrow c_0$ such that:*

$$T = T' \circ q^* + S,$$

$$\|T'\| \leq (4 + \varepsilon')\|T\|, \quad \|S\| \leq (1 + \varepsilon')\|T\| \quad \text{and} \quad \|S \circ R\| \leq \varepsilon.$$

Proof of Theorem 2.4. We first observe that it is sufficient to prove the theorem in the case when K is zero-dimensional. Namely, if K is an arbitrary compact line then $K \times \{0, 1\}$, endowed with the lexicographic order, is a zero-dimensional compact line and we have an isometric embedding π_1^* of $C(K)$ into $C(K \times \{0, 1\})$ induced by the first projection $\pi_1 : K \times \{0, 1\} \rightarrow K$.

In what follows, K is a zero-dimensional compact line and X is a Banach subspace of $C(K)$ having separable dual. Our goal is to show that the restriction operator $\tau : \mathcal{B}(C(K), c_0) \rightarrow \mathcal{B}(X, c_0)$ is onto, where $\mathcal{B}(X_1, X_2)$ denotes the space of bounded operators from X_1 to X_2 . To this aim, we will prove that the closure of the image of the unit ball of $\mathcal{B}(C(K), c_0)$ under τ contains the open ball of $\mathcal{B}(X, c_0)$ of radius $\frac{1}{8}$. The surjectivity of τ will then follow from [3, Lemma 2.23].

Pick L and $q : K \rightarrow L$ as in Lemma 2.6 and denote by $R : X \rightarrow C(L)$ the restriction to X of the isometry $(q^*)^{-1} : q^*C(L) \rightarrow C(L)$. Let $T_0 \in \mathcal{B}(X, c_0)$ be fixed with $\|T_0\| < \frac{1}{8}$ and let $\varepsilon > 0$ be given. Since $C(L)$ is separable, by Sobczyk's Theorem, there exists $T \in \mathcal{B}(C(L), c_0)$ with $\|T\| \leq 2\|T_0\| < \frac{1}{4}$ and $T \circ R = T_0$. Apply Lemma 2.7 with $\varepsilon' > 0$ chosen such that $(4 + \varepsilon')\|T\| \leq 1$. Then $T' \in \mathcal{B}(C(K), c_0)$ satisfies $\|T'\| \leq 1$ and $\|\mathfrak{r}(T') - T_0\| = \|S \circ R\| \leq \varepsilon$. $\square$

Remark 2.8. The proof of Theorem 2.4 actually shows that given a compact line K , a Banach subspace X of $C(K)$ with separable dual, a bounded operator $T : X \rightarrow c_0$, and $\varepsilon > 0$, there exists an extension $T' : C(K) \rightarrow c_0$ of T with $\|T'\| \leq (8 + \varepsilon)\|T\|$. The authors do not know if this estimate on the extension constant can be improved. We note that in [1, Theorem 3.1] we have shown that, under the additional assumption that X is a Banach subalgebra of $C(K)$, the constant 8 can be replaced with 2.

3. PROOF OF MAIN LEMMAS

The goal of this section is to prove Lemma 2.7. This will be achieved by translating the problem of decomposing the operator T as $T = T' \circ q^* + S$ into a problem of decomposing a sequence of measures $(\mu_n)_{n \geq 1}$ as $\mu_n = \mu'_n + \nu_n$. First, we have to understand how to recognize, in terms of sequences of measures, which c_0 -valued operators on $C(L)$ are of the form $T' \circ q^*$, for some $T' : C(K) \rightarrow c_0$. This is the purpose of Lemma 3.2.

For the proof of Lemma 3.2, we need a more concrete representation of $M(K)$ when K is a compact line. Given a compact line K and a map $F : K \rightarrow \mathbb{R}$, the total variation $V(F) \in [0, +\infty]$ is defined exactly as in the case $K = [0, 1]$. We denote by $\text{BV}(K)$ the Banach space of functions $F : K \rightarrow \mathbb{R}$ of bounded variation (i.e., functions F with $V(F) < +\infty$) endowed with the norm:

$$\|F\|_{\text{BV}} = |F(0)| + V(F).$$

Then:

$$\text{NBV}(K) = \{F \in \text{BV}(K) : F \text{ is right-continuous}\}$$

is a closed subspace of $\text{BV}(K)$. The following representation theorem is standard when $K = [0, 1]$ and a proof can be obtained, for instance, using Riemann–Stieltjes integration ([20, 4.32]). The proof in the case of a general compact line K is similar. However, a little more care is required in the definition of the Riemann–Stieltjes sum, because of the possible presence of consecutive points. We give the details below.

Lemma 3.1. *Given a compact line K , the map:*

$$(2) \quad M(K) \ni \mu \mapsto F_\mu \in \text{NBV}(K)$$

is a linear isometry, where F_μ is defined by $F_\mu(t) = \mu([0, t])$, for all $t \in K$.

Proof. Given $\mu \in M(K)$, we have that $\|F_\mu\|_{\text{BV}} \leq \|\mu\|$ and it follows from the regularity of μ that F_μ is right-continuous. Using Riemann–Stieltjes integration, an inverse for (2) can be defined. More precisely, for $f \in C(K)$ and $F \in \text{NBV}(K)$, we define the Riemann–Stieltjes sum $\mathcal{S}(f, F; P)$ by setting:

$$\mathcal{S}(f, F; P) = f(0)F(0) + \sum_{i=0}^{n-1} f(t_{i+1})(F(t_{i+1}) - F(t_i)),$$

where $P : 0 = t_0 < t_1 < \dots < t_n = \max K$ is a partition of K . It is easy to prove that, for $f \in C(K)$ and $\varepsilon > 0$, there exists a partition P of K such that f oscillates less than ε on each interval $[t_i, t_{i+1}]$ of P with $]t_i, t_{i+1}[\neq \emptyset$. Using this fact, it follows that the Riemann–Stieltjes integral is well-defined by $\int_K f dF = \lim_P \mathcal{S}(f, F; P)$, with partitions P of K ordered by inclusion. For $F \in \text{NBV}(K)$, the bounded linear functional $f \mapsto \int_K f dF$ on $C(K)$ is represented by a measure $\mu_F \in M(K)$ with $\|\mu_F\| \leq \|F\|_{\text{BV}}$. By standard arguments, one shows that the operator $\text{NBV}(K) \ni F \mapsto \mu_F \in M(K)$ is the inverse of (2). $\square$

Lemma 3.2. *Let K and L be compact lines and $q : K \rightarrow L$ be a continuous increasing surjection. Assume that K is zero-dimensional. Set:*

$$Q = \{t \in L : |q^{-1}(t)| > 1\},$$

where $|\cdot|$ denotes the cardinality of a set. Let $T : C(L) \rightarrow c_0$ be a bounded operator associated with a weak*-null sequence $(\mu_n)_{n \geq 1}$ in $M(L)$. The following conditions are equivalent:

- (a) *there exists a bounded operator $T' : C(K) \rightarrow c_0$ with $T' \circ q^* = T$;*
- (b) *there exists a countable subset E of Q such that $\mu_n([0, t]) \rightarrow 0$, for all $t \in Q \setminus E$;*
- (c) *there exists a bounded operator $T' : C(K) \rightarrow c_0$ with $T' \circ q^* = T$ and $\|T'\| \leq 2\|T\|$.*

Proof. Assume (a) and let us prove (b). Let T' be associated with a weak*-null sequence $(\mu'_n)_{n \geq 1}$ in $M(K)$. Let $F_n \in \text{NBV}(L)$ and $F'_n \in \text{NBV}(K)$ correspond, respectively, to μ_n and μ'_n as in Lemma 3.1. From $T' \circ q^* = T$ we obtain $q_*(\mu'_n) = \mu_n$, for all $n \geq 1$, where $q_* : M(K) \rightarrow M(L)$ denotes the adjoint of q^* . The operator q_* is given by:

$$q_*(\mu')(B) = \mu'(q^{-1}[B]), \quad \mu' \in M(K),$$

for every Borel subset B of L . It follows that:

$$F_n(t) = F'_n(b_t),$$

for all $t \in L$, where $b_t = \max(q^{-1}(t))$. Since K is zero-dimensional, for each $t \in Q$, there exists a right-isolated point a_t of K in $q^{-1}(t)$. For all $t \in Q$, we have $F'_n(b_t) - F'_n(a_t) = \mu'_n([a_t, b_t])$ and, since the intervals $[a_t, b_t]$ are disjoint, it follows that $F'_n(b_t) - F'_n(a_t) = 0$, for all $t \in Q$ outside a countable subset E_n of Q . Set $E = \bigcup_{n=1}^{\infty} E_n$. For $t \in Q \setminus E$, we have $F_n(t) = F'_n(a_t) \rightarrow 0$, since $[0, a_t]$ is a clopen subset of K and $(\mu'_n)_{n \geq 1}$ is weak*-null.

Now assume (b) and let us prove (c). Define F_n and b_t as above. For each $n \geq 1$, set $G_n = F_n \circ q$. Then $G_n \in \text{NBV}(K)$ and $\|G_n\|_{\text{BV}} = \|F_n\|_{\text{BV}}$, so that G_n corresponds to a measure $\nu_n \in M(K)$ with $\|\nu_n\| = \|\mu_n\|$ and the operator $S : C(K) \rightarrow \ell_\infty$ associated with $(\nu_n)_{n \geq 1}$ satisfies $\|S\| = \|T\|$. Moreover, since $F_n(t) = G_n(b_t)$, for all $t \in L$, we have $S \circ q^* = T$. We will show that the quotient $C(K)/S^{-1}[c_0]$ is separable and it will follow from [1, Proposition 2.2, (a)] that $S|_{S^{-1}[c_0]}$ admits an extension $T' : C(K) \rightarrow c_0$ with $\|T'\| \leq 2\|S\|$. Since K is zero-dimensional, the set:

$$(3) \quad \{\chi_{[0,s]} : s \text{ right-isolated in } K\}$$

spans a dense subspace of $C(K)$. Let us prove that the image of (3) under the quotient map $C(K) \rightarrow C(K)/S^{-1}[c_0]$ is countable. We claim that if $s \in K$ is right-isolated and $s \notin q^{-1}[E]$ then $\chi_{[0,s]} \in S^{-1}[c_0]$. To prove the claim, note first that $S(\chi_{[0,s]}) = (G_n(s))_{n \geq 1}$. In case $q(s) \in Q$, we have $q(s) \in Q \setminus E$ and therefore:

$$(4) \quad G_n(s) = F_n(q(s)) \longrightarrow 0.$$

In case $q(s) \notin Q$, since s is right-isolated in K , we have that $q(s)$ is right-isolated in L and then (4) follows from the fact that $(\mu_n)_{n \geq 1}$ is weak*-null. This proves the claim. To conclude the proof of (c), simply observe that for $t \in E$ and $s_1, s_2 \in q^{-1}(t)$ right-isolated in K , we have $\chi_{[0,s_1]} - \chi_{[0,s_2]} \in \text{Ker}(S) \subset S^{-1}[c_0]$. $\square$

Remark 3.3. In the proof of (a) $\Rightarrow$ (b) in Lemma 3.2, the assumption that K is zero-dimensional is only used to establish that $q^{-1}(t)$ contains a right-isolated point of K , for all $t \in Q$.

We are now ready to state the measure-theoretic version of Lemma 2.7.

Lemma 3.4. *Let L be a metrizable zero-dimensional compact line, X be a Banach space with separable dual, $R : X \rightarrow C(L)$ be a bounded operator and $(\mu_n)_{n \geq 1}$ be a weak*-null sequence in $M(L)$ with $\sup_{n \geq 1} \|\mu_n\| \leq 1$. Given $\varepsilon, \varepsilon' > 0$, there exist weak*-null sequences $(\mu'_n)_{n \geq 1}$ and $(\nu_n)_{n \geq 1}$ in $M(L)$ such that:*

- (a) $\mu_n = \mu'_n + \nu_n$, for all $n \geq 1$;
- (b) $\|\nu_n\| \leq 1 + \varepsilon'$ and $\|\mu'_n\| \leq 2 + \varepsilon'$, for all $n \geq 1$;
- (c) $\|R^*(\nu_n)\| \leq \varepsilon$, for all $n \geq 1$;
- (d) $\mu'_n([0, t]) \longrightarrow 0$, for all t outside a countable set.

We now prove Lemma 2.7 using Lemma 3.4.

Proof of Lemma 2.7. Without loss of generality, assume $\|T\| = 1$. Denote by $(\mu_n)_{n \geq 1}$ the weak*-null sequence in $M(L)$ associated with T . Using Lemma 3.4 we obtain weak*-null sequences $(\mu'_n)_{n \geq 1}$ and $(\nu_n)_{n \geq 1}$ in $M(L)$ satisfying (a)–(d). Let $T'_0 : C(L) \rightarrow c_0$ and $S : C(L) \rightarrow c_0$ be the bounded

operators associated with $(\mu'_n)_{n \geq 1}$ and $(\nu_n)_{n \geq 1}$, respectively. By item (a), $T = T'_0 + S$. Item (b) implies:

$$\|T'_0\| \leq 2 + \varepsilon', \quad \|S\| \leq 1 + \varepsilon',$$

and item (c) gives $\|S \circ R\| \leq \varepsilon$. Finally, Lemma 3.2 and (d) yield a bounded operator $T' : C(K) \rightarrow c_0$ such that $\|T'\| \leq 2\|T'_0\|$ and $T' \circ q^* = T'_0$. $\square$

The proof of Lemma 3.4 is hard and it requires a few more lemmas. Let us first introduce some notation.

Given a Banach space X , a compact Hausdorff space L and a bounded operator $R : X \rightarrow C(L)$, we denote by $\phi^R : L \rightarrow X^*$ the map defined by:

$$\phi^R(p) = R^*(\delta_p), \quad p \in L.$$

Obviously, the map ϕ^R is continuous when X^* is endowed with the weak*-topology and:

$$\|R\| = \sup_{p \in L} \|\phi^R(p)\|.$$

Recall that a Banach space X is said to be *weak*-fragmentable* if for every nonempty bounded subset B of X^* and every $\varepsilon > 0$, there exists a nonempty subset U of B , weak*-open relatively to B , with $\text{diam}(U) < \varepsilon$, where $\text{diam}(U)$ denotes the diameter of U . It is well-known that X is weak*-fragmentable if and only if X is Asplund; in particular, if X has separable dual then X is weak*-fragmentable (see [4, Theorem 11.8]).

Lemma 3.5. *Let X be a Banach space, L be a compact Hausdorff space and $R : X \rightarrow C(L)$ be a bounded operator. Given $\delta > 0$, define by recursion on the ordinal α a decreasing family of closed subsets H_α of L by setting $H_0 = L$,*

$$H_{\alpha+1} = \{p \in H_\alpha : \text{diam}(\phi^R[V]) \geq \delta, \text{ for every } n\text{-hood } V \text{ of } p \text{ in } H_\alpha\},$$

and $H_\alpha = \bigcap_{\beta < \alpha} H_\beta$ if α is a limit ordinal. If X is weak-fragmentable, then there exists α such that $H_\alpha = \emptyset$.*

Proof. The conclusion will follow from the fact that, given an ordinal α with $H_\alpha \neq \emptyset$, the set $H_{\alpha+1}$ is properly contained in H_α . To prove the latter, note that $\phi^R[H_\alpha]$ is a nonempty bounded subset of X^* and thus there exists a nonempty weak*-open subset U of $\phi^R[H_\alpha]$ with $\text{diam}(U) < \delta$. Then $(\phi^R|_{H_\alpha})^{-1}[U]$ is a nonempty open subset of H_α and it is disjoint from $H_{\alpha+1}$. $\square$

Remark 3.6. We note that the hierarchy of closed subsets H_α of L defined in the statement of Lemma 3.5 coincides precisely with the oscillation hierarchy $P_{\delta, \phi^R}^\alpha$ of the map $\phi^R : L \rightarrow X^*$ defined in [8, 1.II] and that the least ordinal α with $H_\alpha = \emptyset$ is the δ -oscillation rank $\beta(\phi^R, \delta)$ of the map ϕ^R defined in that article. Moreover, it is easily checked that such α is bounded by the Szlenk index $\text{Sz}(X)$ of the Banach space X . See [17] for Szlenk's original definition, [11] for a survey, and [12, Proposition 3.3] for the equivalence (in the case

of spaces not containing ℓ_1) between Szlenk's original definition and the definition appearing in [11]. We observe that the Szlenk index of a Banach space is well-defined if and only if the space is Asplund ([11, Theorem 2]) and it is countable if the space has separable dual ([11, Theorem 1]).

Lemma 3.7. *Let X be a Banach space, L be a compact Hausdorff space and $R : X \rightarrow C(L)$ be a bounded operator. If $\mu \in M(L)$ satisfies $\mu(L) = 0$ then:*

$$\|R^*(\mu)\| \leq \frac{1}{2} \text{diam}(\phi^R[\text{supp } \mu]) \|\mu\|.$$

Proof. We can assume without loss of generality that $\text{supp } \mu = L$. Namely, setting $H = \text{supp } \mu$, $\mu' = \mu|_H$, and $R' = \rho_H \circ R$, we have that $\phi^{R'} = \phi^R|_H$, $(R')^*(\mu') = R^*(\mu)$, and $\|\mu\| = \|\mu'\|$, where $\rho_H : C(L) \rightarrow C(H)$ denotes the restriction operator.

Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of the measure μ . Then $\|\mu\| = \mu^+(L) + \mu^-(L)$ and $\mu^+(L) = \mu^-(L)$, so that μ^+ and μ^- are regular nonnegative measures of norm $\frac{1}{2}\|\mu\|$. We can obviously assume that $\|\mu\| = 2$, which implies that μ^+ and μ^- are regular probability measures and therefore they belong to the weak*-closed convex hull of $\{\delta_p : p \in L\}$. Since R^* is weak*-continuous and linear, $R^*(\mu^+)$ and $R^*(\mu^-)$ belong to the weak*-closed convex hull of $\phi^R[L]$. Noting that the diameter of a subset of X^* is equal to the diameter of its weak*-closed convex hull, we obtain:

$$\|R^*(\mu)\| = \|R^*(\mu^+) - R^*(\mu^-)\| \leq \text{diam}(\phi^R[L]),$$

concluding the proof. $\square$

Lemma 3.8. *Let I be a compact line and H be a nonempty closed subset of I . Given $\mu \in M(I)$, there exists $\nu \in M(I)$ such that:*

- (A) $\text{supp } \nu \subset H$;
- (B) $\nu(I) = \mu(I)$;
- (C) $\|\nu\| \leq \|\mu\|$;
- (D) $\nu([\min I, t]) = \mu([\min I, t])$, for all $t \in H \setminus \{\max H\}$.

The assumption $H \neq \emptyset$ can be dropped if $\mu(I) = 0$.

Proof. Let $F_\mu \in \text{NBV}(I)$ be given as in Lemma 3.1. Define $F : I \rightarrow \mathbb{R}$ by setting $F(t) = 0$, for $t < \min H$, $F(t) = F_\mu(\max([0, t] \cap H))$, for $\min H \leq t < \max H$, and $F(t) = F_\mu(\max I)$, for $t \geq \max H$. It is easily checked that $F \in \text{NBV}(I)$ and that $\|F\|_{\text{BV}} \leq \|F_\mu\|_{\text{BV}}$. Then the unique measure $\nu \in M(I)$ with $F_\nu = F$ satisfies the desired conditions. $\square$

To state the next lemma and prove Lemma 3.4, it is convenient to introduce the following terminology. By a *clopen-partition* of a compact line L we mean a finite set P of right-isolated points of L such that $\max L \in P$. If $P = \{b_1, \dots, b_m\}$, with $b_1 < \dots < b_m = \max L$, we write $\bar{P} = \{[0, b_1], [b_1, b_2], \dots, [b_{m-1}, b_m]\}$, so that $\bar{P}$ is a partition of L into clopen intervals. For $t \in L$, we denote by $\bar{P}(t)$ the unique $I \in \bar{P}$ such that $t \in I$.

Lemma 3.9. *Let L be a compact line and P be a clopen-partition of L . Given $\mu \in M(L)$, there exists $\tilde{\mu} \in M(L)$ satisfying the following conditions:*

- (i) $\tilde{\mu}(I) = 0$, for all $I \in \overline{P}$;
- (ii) $\tilde{\mu}([0, t]) = \mu([0, t])$, for all $t \in L \setminus P$;
- (iii) $\|\tilde{\mu}\| \leq \|\mu\| + 2 \sum_{b \in P} |\mu([0, b])|$.

Proof. Let $F_\mu \in \text{NBV}(L)$ be given as in Lemma 3.1. Define $F : L \rightarrow \mathbb{R}$ by setting $F(t) = F_\mu(t)$, for $t \in L \setminus P$, and $F(t) = 0$, for $t \in P$. It is easily checked that $F \in \text{NBV}(L)$ and that $\|F\|_{\text{BV}} \leq \|F_\mu\|_{\text{BV}} + 2 \sum_{b \in P} |\mu([0, b])|$. Then the unique measure $\tilde{\mu} \in M(L)$ with $F_{\tilde{\mu}} = F$ satisfies the desired conditions. $\square$

Proof of Lemma 3.4. Consider the decreasing family of closed subsets H_α of L defined as in Lemma 3.5, with $\delta = \frac{2\varepsilon}{1+\varepsilon'}$. For each clopen interval $I \subset L$, set:

$$(5) \quad \alpha(I) = \min \{ \alpha : \text{diam}(\phi^R[H_\alpha \cap I]) < \delta \}.$$

The ordinal $\alpha(I)$ is well-defined since there exists α with $H_\alpha = \emptyset$ and, for such α , we have $\text{diam}(\phi^R[H_\alpha \cap I]) < \delta$.

Since L is second countable, it has only a countable number of right-isolated points. Let $(P_k)_{k \geq 1}$ be an increasing sequence of clopen-partitions of L such that $\bigcup_{k=1}^\infty P_k$ is equal to the set of all right-isolated points of L . It follows from the fact that L is zero-dimensional that, for each $t \in L$, the set $\{\overline{P_k}(t) : k \geq 1\}$ is a fundamental system of neighborhoods of t . Since $(\mu_n)_{n \geq 1}$ is weak*-null, for each $k \geq 1$ there exists $n_k \geq 1$ such that:

$$(6) \quad 2 \sum_{b \in P_k} |\mu_n([0, b])| \leq \varepsilon',$$

for all $n \geq n_k$. The sequence $(n_k)_{k \geq 1}$ can be chosen to be strictly increasing.

Now let $n \geq 1$ be given and let us define ν_n and μ'_n . If $n < n_1$, set $\nu_n = 0$ and $\mu'_n = \mu_n$, so that (a)—(c) hold. If $n \geq n_1$, pick the only $k \geq 1$ such that $n_k \leq n < n_{k+1}$. Using Lemma 3.9 with $\mu = \mu_n$ and $P = P_k$, we get $\tilde{\mu}_n \in M(L)$ satisfying (i)—(iii). For each $I \in \overline{P_k}$, we have $\tilde{\mu}_n(I) = 0$; apply Lemma 3.8 to the compact line I , the closed subset $H_{\alpha(I)} \cap I$ of I , and the measure $\tilde{\mu}_n|_I$ obtaining $\nu_n^I \in M(I)$ satisfying (A)—(D). Define $\nu_n \in M(L)$ so that $\nu_n|_I = \nu_n^I$, for all $I \in \overline{P_k}$, and set $\mu'_n = \mu_n - \nu_n$. Then (a) holds. The following simple computation proves (b):

$$\begin{aligned} \|\nu_n\| &= \sum_{I \in \overline{P_k}} \|\nu_n^I\| \stackrel{(C)}{\leq} \sum_{I \in \overline{P_k}} \|\tilde{\mu}_n|_I\| = \|\tilde{\mu}_n\| \\ &\stackrel{(iii)}{\leq} \|\mu_n\| + 2 \sum_{b \in P_k} |\mu_n([0, b])| \stackrel{(6)}{\leq} 1 + \varepsilon'. \end{aligned}$$

This yields also $\|\mu'_n\| \leq 2 + \varepsilon'$. To prove (c), note first that:

$$\nu_n = \sum_{I \in \overline{P_k}} \nu_n^I,$$

where $\nu_n^I \in M(I)$ is identified with its extension to L that vanishes identically outside I . By (A), we have $\text{supp } \nu_n^I \subset H_{\alpha(I)} \cap I$; moreover:

$$(7) \quad \nu_n^I(I) \stackrel{(B)}{=} \tilde{\mu}_n(I) \stackrel{(i)}{=} 0.$$

Then, using Lemma 3.7, we obtain:

$$\begin{aligned} \|R^*(\nu_n)\| &\leq \sum_{I \in \overline{P_k}} \|R^*(\nu_n^I)\| \leq \frac{1}{2} \sum_{I \in \overline{P_k}} \text{diam}(\phi^R[H_{\alpha(I)} \cap I]) \|\nu_n^I\| \\ &\stackrel{(5)}{\leq} \frac{1}{2} \delta \|\nu_n\| \leq \varepsilon. \end{aligned}$$

Now let us prove that $(\nu_n)_{n \geq 1}$ (and hence $(\mu'_n)_{n \geq 1}$) is weak*-null. Since L is zero-dimensional and $(\nu_n)_{n \geq 1}$ is bounded, it suffices to fix a right-isolated t in L and check that $\nu_n([0, t]) \rightarrow 0$. Take $k_0 \geq 1$ with $t \in P_{k_0}$ and let us show that $\nu_n([0, t]) = 0$ for $n \geq n_{k_0}$. If $n \geq n_{k_0}$, we have $n_k \leq n < n_{k+1}$ for some $k \geq k_0$. Then $t \in P_k$ and $[0, t]$ is a disjoint union of elements of $\overline{P_k}$. Using (7), we obtain $\nu_n([0, t]) = 0$.

Finally, let us prove (d). Set:

$$E = \{\max(H_{\alpha(I)} \cap I) : I \in \overline{P_k}, k \geq 1 \text{ and } H_{\alpha(I)} \cap I \neq \emptyset\},$$

so that E is a countable subset of L . Let $t \in L \setminus E$ be fixed. We claim that $\mu'_n([0, t]) \rightarrow 0$. Let β be the largest ordinal with $t \in H_\beta$. Then $t \notin H_{\beta+1}$ and thus there exists a neighborhood V of t in H_β with $\text{diam}(\phi^R[V]) < \delta$. Since $\{\overline{P_k}(t) : k \geq 1\}$ is a fundamental system of neighborhoods of t in L , there exists $k_0 \geq 1$ with $H_\beta \cap \overline{P_{k_0}}(t) \subset V$. The claim will be proven once we have established that $\mu'_n([0, t]) = 0$ for $n \geq n_{k_0}$. Given $n \geq n_{k_0}$, there exists $k \geq k_0$ with $n_k \leq n < n_{k+1}$. Setting $I_t = \overline{P_k}(t)$, we have $I_t \subset \overline{P_{k_0}}(t)$, so that $H_\beta \cap I_t \subset V$ and $\text{diam}(\phi^R[H_\beta \cap I_t]) < \delta$. It follows that $\alpha(I_t) \leq \beta$ and thus $t \in H_{\alpha(I_t)} \cap I_t$. Since $t \notin E$, we have that $t \neq \max(H_{\alpha(I_t)} \cap I_t)$; in particular $t \neq \max I_t$ and thus $t \notin P_k$. We compute:

$$\nu_n([0, t]) \stackrel{(7)}{=} \nu_n^{I_t}([\min I_t, t]) \stackrel{(D)}{=} \tilde{\mu}_n([\min I_t, t]) \stackrel{(i)}{=} \tilde{\mu}_n([0, t]) \stackrel{(ii)}{=} \mu_n([0, t]).$$

Hence $\mu'_n([0, t]) = 0$. This proves the claim and concludes the proof. $\square$

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