Twisted sums for scattered spaces

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This project was partially supported by FAPESP grant 2018/09797-2

47th Winter School in Abstract Analysis,
Svratka, Czech Republic, 12-19 January 2019

Definition

Let X and Y be Banach spaces. A twisted sum of Y and X *is a short exact sequence of the form:*

$$0 \longrightarrow Y \xrightarrow{T} Z \xrightarrow{S} X \longrightarrow 0,$$

where Z is a Banach space and the maps T and S are linear and bounded.

Remark

Note that since $T[Y] = KerS$, it follows from the Open Mapping Theorem that Y is isomorphic to $T[Y]$ and the quotient $Z/T[Y]$ is isomorphic to X, through $\overline{S} : Z/T[Y] \to X$.

Example

If X and Y are Banach spaces and the direct sum $Y \bigoplus X$ is endowed with some product norm, then:

$$0 \longrightarrow Y \xrightarrow{i_1} Y \bigoplus X \xrightarrow{\pi_2} X \longrightarrow 0$$

is a twisted sum of Y and X, where i_1 is the canonical embedding and π_2 is the second projection.

Definition

A twisted sum:

$$0 \longrightarrow Y \xrightarrow{T} Z \xrightarrow{S} X \longrightarrow 0$$

of Banach spaces Y and X is called trivial *if $T[Y]$ is complemented in Z.*

Question

Are there nontrivial twisted sums of Banach spaces?

Answer: Yes.

Theorem (Phillips–1940)

The sequence space c_0 is not a complemented subspace of ℓ_∞.

Corollary

The twisted sum:

$$0 \longrightarrow c_0 \xrightarrow{inc} \ell_\infty \xrightarrow{q} \ell_\infty / c_0 \longrightarrow 0,$$

is not trivial, where inc denotes the inclusion map and q denotes the quotient map.

Theorem (Sobczyk–1941)

Every isomorphic copy of c_0 inside a separable Banach space is complemented.

Corollary

If X is a separable Banach space, then every twisted sum of c_0 and X is trivial.

Proof. Let Z be a Banach space such that:

$$0 \longrightarrow c_0 \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is an exact sequence. In this case Z is separable and therefore this twisted sum is trivial. □

Definition

Given a compact Hausdorff space K, we denote by $C(K)$ the Banach space of continuous real-valued functions defined on K, endowed with the supremum norm.

Proposition

Let K be a compact Hausdorff space. The Banach space $C(K)$ is separable if and only if K is metrizable.

Corollary (Corollary of Sobczyk's Theorem)

If K is a metrizable compact space, then every twisted sum of c_0 and $C(K)$ is trivial.

Open Problem (Cabelo, Castillo, Kalton and Yost–2003)

Is there a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial?

This problems remains open, but we are working on it!

Definition

A topological space is said to be scattered if every nonempty subspace has an isolated point with respect to the subspace topology.

Example

If Γ is a discrete topological space, then its one-point compactification $\Gamma \cup \{\infty\}$ is scattered.

Definition

Let $\mathcal{X}$ be a topological space. We define by recursion on α a decreasing family of closed subsets of $\mathcal{X}$:

- $\mathcal{X}^{(0)} = \mathcal{X}$;
- *For every ordinal α, $\mathcal{X}^{(\alpha+1)} = \mathcal{X}^{(\alpha)} \setminus Is(\mathcal{X}^{(\alpha)})$, where $Is(\mathcal{X}^{(\alpha)})$ denotes the set of isolated points of $X^{(\alpha)}$;*
- *For every limit ordinal α, $\mathcal{X}^{(\alpha)} = \bigcap_{\beta\in\alpha} \mathcal{X}^{(\beta)}$.*

The space $\mathcal{X}^{(\alpha)}$ is called the α^{th} Cantor–Bendixson derivative of $\mathcal{X}$

Example

If Γ is an infinite discrete topological space and $\mathcal{X}$ is its one-point compactification, then $\mathcal{X}^{(1)} = \{\infty\}$ and $\mathcal{X}^{(2)} = \emptyset$.

Proposition

A topological space $\mathcal{X}$ is scattered if and only if there exists an ordinal α such that $\mathcal{X}^{(\alpha)} = \emptyset$.

Definition

If $\mathcal{X}$ is scattered, then the height of $\mathcal{X}$ is defined as the least ordinal α such that $\mathcal{X}^{(\alpha)} = \emptyset$. If the height of $\mathcal{X}$ is a natural number, then we say that $\mathcal{X}$ has finite height.

Theorem (Castillo, Top. Appl.–2016)

Assume CH. If K is a nonmetrizable finite height compact space, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Theorem (Marciszewski and Plebanek, JFA–2018)

Assume MA$+\neg$ CH. If K is a separable scattered compact space with height 3 and weight smaller than $\mathfrak{c}$, then every twisted sum of c_0 and $C(K)$ is trivial.

Theorem (Correa and Tausk, Fund. Math.–2018)

Assume MA$+\neg$ CH. If K is a separable scattered compact space with finite height and weight smaller than $\mathfrak{c}$, then every twisted sum of c_0 and $C(K)$ is trivial.

Corollary

It is consistent with ZFC the existence of a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial.

Open Problem

Is there in ZFC a nonmetrizable compact Hausdorff space K such that every twisted sum of c_0 and $C(K)$ is trivial?

Question

What happens, under MA+$\neg$ CH, if K is a finite height space with big weight, i.e., $w(K) \geq \mathfrak{c}$?

Theorem (Correa–2018)

Assume MA+$\neg$ CH. If K is a finite height compact space with $w(K) \geq \mathfrak{c}$, *then there exists a nontrivial twisted sum of* c_0 *and* $C(K)$.

Open Problem

Does it hold in ZFC that if K is a finite height compact space with $w(K) \geq c$, *then there exists a nontrivial twisted sum of* c_0 *and* $C(K)$?

Question

What happens, under MA+$\neg$ CH, if K is a nonseparable scattered compact space with $w(K) < \mathfrak{c}$?

Theorem (Marciszewski and Plebanek, JFA–2018)

If K is a nonseparable scattered space with $w(K) = \omega_1$, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

Theorem (Correa–2018)

Assume MA+$\neg$ CH. If K is a nonseparable scattered space $w(K) < \mathfrak{c}$, then there exists a nontrivial twisted sum of c_0 and $C(K)$.

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