On the c_0-extension property

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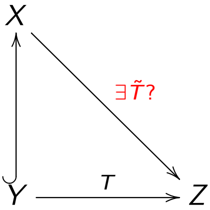
Question

Let X and Z be Banach spaces, Y be a closed subspace of X and $T : Y \to Z$ be a bounded operator. Does T admit a bounded and linear extension $\tilde{T} : X \to Z$?



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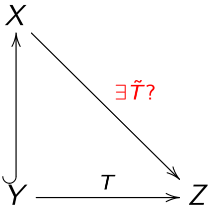
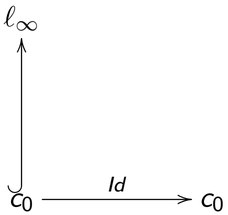


Answer: No.

Theorem (Phillips–1940)

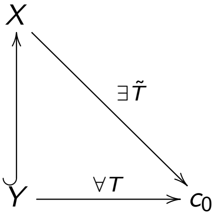
The identity operator of c_0 does not admit a bounded and linear extension defined on ℓ_∞.



Sobczyk's theorem

Theorem (Sobczyk–1941)

If X is a **separable** *Banach space, then every c_0-valued bounded operator defined on a closed subspace of X admits a bounded and linear extension defined on X.*



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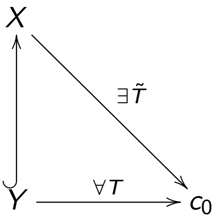

The heart of Sobczyk's theorem is the weak-star metrizability of the closed dual unit ball of the separable Banach space X.

Notation

Given a Banach space X, we write:

$$B_{X^*} = \{\alpha \in X^* : \|\alpha\| \leq 1\}.$$

We denote by w^* the weak-star topology on X^*.

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Proposition

A Banach space X is separable if and only if B_{X^*} is w^*-metrizable.

Definition (Claudia and Daniel–2014)

*We say that a Banach space X has the c_0-***extension property*** (c_0-EP) if every c_0-valued bounded operator defined on a closed subspace of X admits a c_0-valued bounded extension defined on X.*

The c_0-extension property

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Example

- Every separable Banach space has the c_0-EP.

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Is there a nonseparable Banach space with the c_0-EP?

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Is there a nonseparable Banach space with the c_0-EP?

Answer: Yes. Every Hilbert space has the c_0-EP.

WCG Banach spaces

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We say that a Banach space is **weakly compactly generated** *(WCG) if it contains a weakly compact subset that is linearly dense.*

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Theorem

Every WCG Banach space has the c_0*-EP.*


WLD spaces are amazing!!
Every WCG space is WLD.

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Remark

Recall that a compact space is metrizable if and only if it embeds homeomorphically in $\mathbb{R}^\omega$.

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We say that a Banach space X is **weakly Lindelöf determined** *(WLD) if (B_{X^*}, w^*) is a Corson compactum.*

Remark

Recall that a compact space is metrizable if and only if it embeds homeomorphically in $\mathbb{R}^\omega$.

Definition

We say that a compact space K is a **Corson compactum** *if there exists a set I such that K embeds homeomorphically in*

$$\Sigma(I) = \{f \in R^I : supp(f) \text{ is countable}\}.$$

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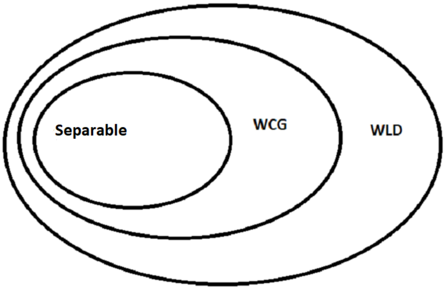

Going beyond WLD spaces

Proposition (Claudia and Daniel–2014)

The space $C[0, \omega_1]$ *has the* c_0*-EP.*

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Proposition (Claudia and Daniel–2014)

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Answer: No.

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What does the space $C[0, \omega_1]$ have in common with the WLD spaces?

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Answer: No.

Question

What does the space $C[0,\omega_1]$ have in common with the WLD spaces?

Answer: If $X = C[0,\omega_1]$ or X is WLD, then (B_{X^*}, w^*) is monolithic.

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Let X be a Banach space. If (B_{X^*}, w^*) is monolithic, then X has the c_0-EP?

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Open Problem

Let K be a compact and Hausdorff space. When $\left(B_{C(K)^*}, w^*\right)$ is monolithic?

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Definition

We say that a compact and Hausdorff space K has **Property (M)** *if every measure in $M(K)$ has separable support.*

The amazing $C(K)$ world

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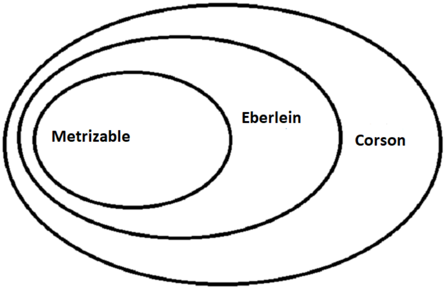

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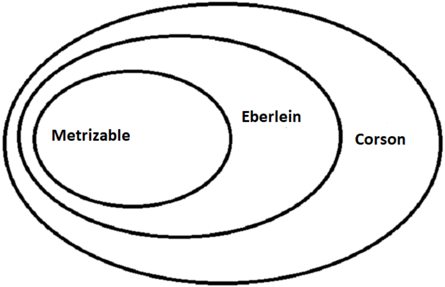

Question

If K is a Corson compactum, then $C(K)$ has the c_0-EP?

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Theorem (Argyros, Mercourakis and Negrepontis–1988)

Assume $MA + \neg CH$. Every Corson compactum has Property (M).

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Theorem (Argyros, Mercourakis and Negrepontis–1988)

Assume CH. There exists a Corson compactum without Property (M).

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Theorem (Claudia–2019)

If K is the Corson compact space built by Argyros, Mercourakis and Negrepontis under CH, then $C(K)$ does not have the c_0-EP.

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Corollary

The existence of a Corson compactum K such that $C(K)$ does not have the c_0-EP is independent from the axioms of ZFC.

Scattered spaces

Proposition

Every scattered compact space has Property (M).

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Open Problem

If K is a scattered compact space such that $C(K)$ has the c_0-EP, then K is monolithic?

Theorem (Claudia–2019)

Let K be a scattered compact space with height at most $\omega + 1$. If $C(K)$ has the c_0-EP, then K is monolithic.


Thank you!

A. Argyros, S. Mercourakis, and S. Negrepontis.
Functional-analytic properties of corson-compact spaces.
Studia Math., 89:197–229, 1988.

C. Correa.
On the c_0-extension property.
https://arxiv.org/pdf/1912.08564.pdf.

C. Correa and D. V. Tausk.
Compact lines and the sobczyk property.
J. Func. Anal., 266 (9):5765–5778, 2014.