ON EXTENSIONS OF c_0-VALUED OPERATORS

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Abstract. We study pairs of Banach spaces (X, Y), with $Y \subset X$, for which the thesis of Sobczyk's theorem holds, namely, such that every bounded c_0-valued operator defined in Y extends to X. We are mainly concerned with the case when X is a $C(K)$ space and $Y \equiv C(L)$ is a Banach subalgebra of $C(K)$. The main result of the article states that, if K is a compact line and L is countable, then every bounded c_0-valued operator defined in $C(L)$ extends to $C(K)$.

1. Introduction

In this article we introduce a definition that is relevant for the investigation of two classical problems in the theory of Banach spaces: the problem of extension of bounded operators and the problem of complementation of copies of c_0 in Banach spaces. Given a Banach space X, we will say that a closed subspace Y of X has the *c_0-extension property* (briefly: c_0EP) in X if every bounded operator $T : Y \to c_0$ admits a bounded (c_0-valued) extension to X. Of course, Y has the c_0EP in X if, for example, Y is complemented in X. Our main result (Theorem 3.1) concerns the c_0EP for certain Banach subalgebras of a space $C(K)$. More precisely, it states that a Banach subalgebra $C(L)$ of $C(K)$ always has the c_0EP in $C(K)$ when L is countable and K is a compact line. By a *compact line* we mean a linearly ordered set which is compact in the order topology. The problem of complementation of Banach subalgebras of $C(K)$ spaces, with K a compact line, was studied in a recent article [5].

Recall that ℓ_∞ is an *injective* Banach space, i.e., for any Banach space X and any closed subspace Y of X, every bounded operator $T : Y \to \ell_\infty$ admits a bounded extension to X. In particular, every isomorphic copy of ℓ_∞ in a Banach space is complemented. It is well-known [9] that the space c_0 is not injective: namely, c_0 is not complemented in ℓ_∞. However, the celebrated theorem of Sobczyk [12], says that c_0 is *separably injective*, i.e., if X is a separable Banach space and Y is a closed subspace of X then every bounded operator $T : Y \to c_0$ admits a bounded extension to X. Using our terminology, this means that every closed subspace of a separable Banach

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space X has the c_0EP in X. In particular, every isomorphic copy of c_0 in a separable Banach space is complemented. A Banach space in which every isomorphic copy of c_0 is complemented is often called a *Sobczyk's space*.

There are many possible directions in which one can search for generalizations of Sobczyk's theorem: for instance, [7, 8] one can look for classes of nonseparable Banach spaces in which every isomorphic (or every isometric) copy of c_0 is complemented. One can also [2] replace the space c_0 with $c_0(\Gamma)$, where Γ is an uncountable set, or [11] replace c_0 with its vector-valued versions (i.e., c_0-type sums of Banach spaces).

Our approach is to investigate pairs of Banach spaces (X, Y), with $Y \subset X$, for which the thesis of Sobczyk's theorem holds. For instance, a simple adaptation (Proposition 2.2) of Veech's proof of Sobczyk's theorem [13] shows that Y has the c_0EP in X if X/Y is separable or if X is weakly compactly generated (WCG). We will say that X has the (resp., *separable*) c_0*-extension property* if every (resp., separable) closed subspace of X has the c_0EP in X. Thus, every WCG space has the c_0EP. Note that if a separable Banach subspace Y of a Banach space X has the c_0EP in X then every isomorphic copy of c_0 in Y is complemented in X. In particular, if X has the separable c_0EP then X is Sobczyk.

Since c_0-valued bounded operators can be identified with weak*-null sequences of linear functionals, we have that a Banach subspace Y of X has the c_0EP in X if and only if every weak*-null sequence $(\alpha_n)_{n\geq 1}$ in Y^* extends to a weak*-null sequence $(\tilde{\alpha}_n)_{n\geq 1}$ in X^* (i.e., $\tilde{\alpha}_n$ extends α_n, for all $n \geq 1$). With such formulation in terms of linear functionals, the c_0EP for certain given Banach spaces can be investigated using concrete representations of their dual spaces and of weak*-convergence in those dual spaces (see proof of Proposition 2.9).

As mentioned earlier, we are particularly interested in studying the c_0EP for spaces of the form $C(K)$. (As usual, $C(K)$ denotes the Banach space of real-valued continuous functions on the compact Hausdorff space K, endowed with the supremum norm.) In order to study the question of whether a separable Banach subspace X of a $C(K)$ space has the c_0EP in $C(K)$, one can look at the (separable) Banach subalgebra spanned by X: namely, if such Banach subalgebra has the c_0EP in $C(K)$ then so does X. In particular, $C(K)$ has the separable c_0EP if and only if every separable Banach subalgebra of $C(K)$ has the c_0EP in $C(K)$. Thus, we are mainly interested in understanding the c_0EP for separable Banach subalgebras in $C(K)$ spaces.

Given compact Hausdorff spaces K, L and a continuous map $\phi : K \to L$, we denote by $\phi^* : C(L) \to C(K)$ the composition map $f \mapsto f \circ \phi$. The map ϕ^* is a Banach algebra homomorphism; if ϕ is onto then ϕ^* is also an isometric embedding and therefore the range $\phi^* C(L)$ of ϕ^* can be identified with $C(L)$. Recall that all Banach subalgebras (with unity) of $C(K)$ are of the form $\phi^* C(L)$, for some continuous surjective map $\phi : K \to L$.

In [5, Lemma 2.7] it is given a necessary and sufficient condition for the complementation of $\phi^*C(L)$ in $C(K)$, if both K and L are compact lines and $\phi : K \to L$ is a continuous *increasing* surjection. It would be interesting to have a characterization of Banach subalgebras $\phi^*C(L)$ of $C(K)$, with K a compact line, having the c_0EP in $C(K)$. Our main result Theorem 3.1 is a step in this direction: we establish that $\phi^*C(L)$ always has the c_0EP in $C(K)$ if L is countable. We note that, under the same assumptions, it is not always true that $\phi^*C(L)$ is complemented in $C(K)$, even when ϕ is increasing (Example 2.12).

In Theorem 3.1, we do not have to assume that the map ϕ be increasing. In fact, under the assumption that ϕ be increasing, the proof that $\phi^*C(L)$ has the c_0EP in $C(K)$ is much simpler and we state this weaker result as Corollary 2.6. We observe that the assumption that L be countable cannot be replaced with the weaker assumption that L be metrizable: namely, in Proposition 2.9, we give an example of a continuous increasing surjection $\phi : K \to L$, with L metrizable, such that $\phi^*C(L)$ does not have the c_0EP in $C(K)$.

The key result used in the proof of Theorem 3.1 is Theorem 3.2 which states, roughly, that the class of spaces L for which the thesis of Theorem 3.1 holds is closed under the operation of taking the Alexandroff compactification of topological sums. It turns out that, using Theorem 3.2, one can obtain a stronger version of Theorem 3.1: namely, the assumption that L be countable can be replaced with the (weaker) assumption that L be scattered and hereditarily paracompact (Remark 3.6).

The article is organized as follows. In Section 2 we prove a slight generalization of Sobczyk's theorem (Proposition 2.2) and a technical lemma (Lemma 2.3) that will be used in Section 3. In Subsection 2.1, we focus on Banach spaces of the form $C(K)$, giving a sufficient condition for a subspace to have the c_0EP in $C(K)$ (Lemma 2.5). Finally, Section 3 is devoted to the proof of the main result of this article (Theorem 3.1). The proof of Theorem 3.2, which is the hard work, is presented in Subsection 3.1.

2. The c_0-extension property

In what follows, X and Y always denote (real) Banach spaces. We denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators $T : X \to Y$.

Definition 2.1. Given $\lambda \geq 1$, we say that a closed subspace Y of X has the c_0-*extension property with constant* λ (briefly: λ-c_0EP) in X if every bounded operator $T \in \mathcal{B}(Y, c_0)$ admits an extension $T' \in \mathcal{B}(X, c_0)$ with $\|T'\| \leq \lambda\|T\|$. We say that X has the (resp., *separable*) c_0-*extension property with constant* λ if every (resp., separable) closed subspace of X has the λ-c_0EP in X.

We note that if Y has the c_0EP in X then Y has the λ-c_0EP in X for some $\lambda \geq 1$. Namely, if Y has the c_0EP in X then the restriction map $r : \mathcal{B}(X, c_0) \to \mathcal{B}(Y, c_0)$ is onto and thus it induces an isomorphism $\bar{r} : \mathcal{B}(X, c_0)/\operatorname{Ker}(r) \to \mathcal{B}(Y, c_0)$; any $\lambda > \|\bar{r}^{-1}\|$ does the job.

Recall that a Banach space X is *weakly compactly generated* (WCG) if it admits a weakly compact subset K that is linearly dense in X, i.e., the span of K is dense in X. The following generalization of Sobczyk's theorem is folkloric, but we present the proof for the sake of completeness.

Proposition 2.2. *Let X be a Banach space. Then:*

(a) *if Y is a closed subspace of X such that X/Y is separable then Y has the 2-c_0EP in X;*
(b) *if X is WCG then X has the 2-c_0EP.*

Proof. The proof of (a) is a simple adaptation of the proof of Sobczyk's theorem given in [1, pg. 47], using a translation invariant pseudo-metric ρ on X^* that induces the topology of convergence at the points of a countable subset $C \subset X$ such that $C + Y$ is dense in X.

To prove (b), let Y be a closed subspace of X, $T : Y \to c_0$ be a bounded operator, and $S : X \to \ell_\infty$ be an ℓ_∞-valued extension of T with $\|S\| = \|T\|$. Since $X/\operatorname{Ker}(S)$ admits a bounded linear injection to ℓ_∞, it has a weak*-separable dual; then $X/\operatorname{Ker}(S)$ is separable, because it is a WCG space with a weak*-separable dual. In particular, $X/S^{-1}[c_0]$ is separable and, by (a), the map $S|_{S^{-1}[c_0]}$ has a c_0-valued extension $T' : X \to c_0$ with $\|T'\| \le 2\|S\|$. Clearly, T' extends T and $\|T'\| \le 2\|T\|$. $\square$

Note that having the λ-c_0EP is a hereditary property (yet WCG is not [10]) and thus Proposition 2.2 implies that every Banach subspace of a WCG Banach space has the 2-c_0EP. Also, in view of Proposition 2.2, it is natural to ask whether Y has the c_0EP in X if X/Y is WCG. We will see in Remark 2.10 below that the answer is negative.

Clearly, a closed subspace Y of X has the λ-c_0EP in X if and only if every weak*-null sequence $(\alpha_n)_{n\ge1}$ in Y^* extends to a weak*-null sequence $(\tilde{\alpha}_n)_{n\ge1}$ in X^* with $\sup_{n\ge1} \|\tilde{\alpha}_n\| \le \lambda \sup_{n\ge1} \|\alpha_n\|$. In the next lemma, we show that, if Y has the λ-c_0EP in X, then the weak*-null extension $(\tilde{\alpha}_n)_{n\ge1}$ can in fact be chosen with $\|\tilde{\alpha}_n\| \le \lambda\|\alpha_n\|$, for each n. Moreover, in order to establish that Y has the λ-c_0EP in X, it is sufficient to consider weak*-null sequences $(\alpha_n)_{n\ge1}$ in Y^* that are normalized. We say that a family in a Banach space is *normalized* if all of its members have norm 1.

Lemma 2.3. *Let X be a Banach space, Y be a closed subspace of X and $\lambda \ge 1$. The following statements are equivalent:*

(a) *Y has the λ-c_0EP in X;*
(b) *every normalized weak*-null sequence $(\alpha_n)_{n\ge1}$ in Y^* extends to a weak*-null sequence $(\tilde{\alpha}_n)_{n\ge1}$ in X^* with $\|\tilde{\alpha}_n\| \le \lambda$, for all $n \ge 1$;*
(c) *every weak*-null sequence $(\alpha_n)_{n\ge1}$ in Y^* extends to a weak*-null sequence $(\tilde{\alpha}_n)_{n\ge1}$ in X^* with $\|\tilde{\alpha}_n\| \le \lambda\|\alpha_n\|$, for all $n \ge 1$.*

Proof. The implications (a)⇒(b) and (c)⇒(a) are trivial, so it remains to prove (b)⇒(c). Let $(\alpha_n)_{n\geq 1}$ be a weak*-null sequence in Y^*. Without loss of generality, assume $\sup_{n\geq 1}\|\alpha_n\| = 1$. Set $\beta_n = \frac{\alpha_n}{\|\alpha_n\|}$ when $\alpha_n \neq 0$. For each integer $k \geq 1$, let N_k denote the set of all integers $n \geq 1$ such that $\frac{1}{k+1} < \|\alpha_n\| \leq \frac{1}{k}$. If, for a given k, the set N_k is infinite, then $(\beta_n)_{n\in N_k}$ is a normalized weak*-null sequence in Y^*; by (b), it extends to a weak*-null sequence $(\tilde{\beta}_n)_{n\in N_k}$ in X^* with $\|\tilde{\beta}_n\| \leq \lambda$, for all $n \in N_k$.

We now define the sequence $(\tilde{\alpha}_n)_{n\geq 1}$ in X^* as follows. If $\alpha_n = 0$, set $\tilde{\alpha}_n = 0$. If $n \in N_k$ and N_k is finite, let $\tilde{\alpha}_n \in X^*$ be a Hahn–Banach extension of α_n. If $n \in N_k$ and N_k is infinite, set $\tilde{\alpha}_n = \|\alpha_n\|\tilde{\beta}_n$. Clearly, $\tilde{\alpha}_n$ extends α_n and $\|\tilde{\alpha}_n\| \leq \lambda\|\alpha_n\|$, for all $n \geq 1$. Moreover, if N_k is infinite, $(\tilde{\alpha}_n)_{n\in N_k}$ is weak*-null. Let us show that $(\tilde{\alpha}_n)_{n\geq 1}$ is weak*-null. Fix $x \in X$ with $\|x\| = 1$ and let $\varepsilon > 0$ be given. Choose $k_0 \geq 1$ with $\frac{\lambda}{k_0} < \varepsilon$. For $k \geq k_0$ and $n \in N_k$, we have $|\tilde{\alpha}_n(x)| \leq \lambda\|\alpha_n\| < \varepsilon$. Clearly, for $n \in \bigcup_{k<k_0} N_k$ sufficiently large, we have $|\tilde{\alpha}_n(x)| < \varepsilon$. This concludes the proof. □

Corollary 2.4. *Let X be a Banach space, Y be a closed subspace of X and $\lambda \geq 1$. Assume that every normalized weak*-null sequence $(\alpha_n)_{n\geq 1}$ in Y^* extends to a weak*-null sequence $(\tilde{\alpha}_n)_{n\geq 1}$ in X^* such that:*

$$\limsup_{n\to+\infty} \|\tilde{\alpha}_n\| \leq \lambda. \tag{1}$$

Then, for any $\varepsilon > 0$, Y has the $(\lambda+\varepsilon)$-c_0EP in X.

Proof. Inequality (1) implies that $\|\tilde{\alpha}_n\| \leq \lambda+\varepsilon$, for all n greater than some n_0. Now note that we can replace $\tilde{\alpha}_n$ with a Hahn–Banach extension of α_n, for $n \leq n_0$. □

2.1. **The $\mathbf{c_0}$-extension property for $\mathbf{C(K)}$ spaces.** In what follows, K and L denote compact Hausdorff spaces. We always identify the dual space of $C(K)$ with the space $M(K)$ of finite countably-additive signed regular Borel measures on K, endowed with the total variation norm $\|\mu\| = |\mu|(K)$. Given a point $p \in K$, we denote by $\delta_p \in M(K)$ the probability measure with support $\{p\}$. Note that if $\phi : K \to L$ is a continuous map then the adjoint of the composition map $\phi^* : C(L) \to C(K)$ is the push-forward map $\phi_* : M(K) \to M(L)$ defined by $\phi_*(\mu)(B) = \mu\big(\phi^{-1}[B]\big)$, for any $\mu \in M(K)$ and any Borel subset B of L. When ϕ is onto, we can identify $C(L)$ with the subspace $\phi^*C(L)$ of $C(K)$ and then an extension of $\mu \in M(L) \equiv C(L)^*$ to $C(K)$ is identified with a measure $\tilde{\mu} \in M(K)$ such that $\phi_*\tilde{\mu} = \mu$.

The following lemma gives a simple positive criterion for a Banach subspace of $C(K)$ to have c_0EP in $C(K)$. Recall that K is said to be an *Eberlein compact* if K is homeomorphic to a weakly compact subset of some Banach space. Moreover, K is Eberlein if and only if $C(K)$ is WCG.

Lemma 2.5. *Let K, L be compact Hausdorff spaces, $\phi : K \to L$ be a continuous map and assume that there exists an Eberlein compact subset F*

(for instance, a metrizable closed subset) of K such that $\phi|_F$ is onto. Then every closed subspace of $\phi^*C(L)$ has the 2-c_0EP in $C(K)$.

Proof. Consider a closed subspace $\phi^*[X]$ of $\phi^*C(L)$, where X is a closed subspace of $C(L)$. We have to show that for any $T \in \mathcal{B}(X, c_0)$, there exists $T' \in \mathcal{B}\big(C(K), c_0\big)$ with $T' \circ \phi^*|_X = T$ and $\|T'\| \leq 2\|T\|$. Since $\phi|_F$ is onto, $(\phi|_F)^*$ embeds X isometrically into $C(F)$; by Proposition 2.2, since $C(F)$ is WCG, for any $T \in \mathcal{B}(X, c_0)$ there exists $T_1 \in \mathcal{B}\big(C(F), c_0\big)$ such that $T_1 \circ (\phi|_F)^*|_X = T$ and $\|T_1\| \leq 2\|T\|$. To conclude the proof, set $T' = T_1 \circ \rho_F$, where $\rho_F : C(K) \to C(F)$ denotes the restriction map. □

The following corollary is weaker than our main result Theorem 3.1 (except for the fact that we get 2-c_0EP in the thesis of the corollary and $(2+\varepsilon)$-c_0EP in the thesis of Theorem 3.1). However, since its proof is much simpler, we believe it's interesting to state it here.

Corollary 2.6. *Let K be a compact line, L be a countable compact Hausdorff space and $\phi : K \to L$ be a continuous surjection. Assume that, for each $p \in L$, $\phi^{-1}(p)$ is a countable union of intervals of K (this is the case, for instance, if L is a compact line and ϕ is increasing). Then $\phi^*C(L)$ has the 2-c_0EP in $C(K)$.*

Proof. For each $p \in L$, $\phi^{-1}(p)$ can be written as a countable disjoint union of closed intervals of K (namely, the convex components of $\phi^{-1}(p)$); denote by F_p the set of all endpoints of such closed intervals and set $F = \bigcup_{p \in L} F_p$. Obviously, F is countable and $\phi|_F$ is onto. Moreover, F is closed, since $K \setminus F$ is a union of open intervals of K. □

Using Lemma 2.5, we easily obtain a class of compact spaces K such that the space $C(K)$ has the separable c_0EP (and is, in particular, Sobczyk). Recall that K is $\aleph_0$*-monolithic* if every separable subspace of K is second countable.

Corollary 2.7. *If K is $\aleph_0$-monolithic then $C(K)$ has the separable 2-c_0EP.*

Proof. A closed separable subspace of $C(K)$ spans a separable Banach subalgebra $\phi^*C(L)$ of $C(K)$, where $\phi : K \to L$ is continuous onto and L is metrizable. Let F be the closure of a countable subset of K that is mapped by ϕ onto a dense subset of L. Then F is metrizable and we are done. □

Remark 2.8. We observe that, for example, every scattered compact line is $\aleph_0$-monolithic. Namely, if K is a scattered compact line and F were a separable nonmetrizable closed subset of K, then F would be a separable nonmetrizable compact line and therefore there would exist a continuous surjection $F \to [0, 1]$ ([5, Proof of Lemma 2.5]). This contradicts the fact that F is scattered.

Recall that the *double arrow space* is the set $\mathrm{DA} = [0, 1] \times \{0, 1\}$ endowed with the lexicographic order and the order topology. The double arrow space is a compact line and the first projection $\pi_1 : \mathrm{DA} \to [0, 1]$ is a continuous increasing surjection. It has long been known ([4, Example 2], [6, pg. 6]) that $\pi_1^* C[0,1]$ is not complemented in $C(\mathrm{DA})$. In Proposition 2.9 below we obtain a new proof of this fact, illustrating that the notion of c_0-extension property provides a technique for proving that a subspace of a Banach space is not complemented. Proposition 2.9 also shows that the thesis of Corollary 2.6 does not hold if one only assumes that L be metrizable (even in the case when L is a compact line and ϕ is increasing).

Proposition 2.9. *The subalgebra $\pi_1^* C[0,1]$ does not have the c_0EP in the Banach space $C(\mathrm{DA})$.*

Proof. Given $\mu \in M(\mathrm{DA})$, $\nu \in M[0,1]$, we define maps $F_\mu : [0,1] \to \mathbb{R}$ and $G_\nu : [0,1] \to \mathbb{R}$, by setting:

$$F_\mu(t) = \mu[(0,0),(t,0)], \quad G_\nu(t) = \nu[0,t], \qquad t \in [0,1].$$

The maps F_μ and G_ν are differences of increasing functions and therefore have bounded variation; in particular, they have at most a countable number of discontinuity points. If $\nu = (\pi_1)_*\mu$, then $G_\nu(t) = F_\mu(t^+)$, for all $t \in [0,1[$. In particular, F_μ and G_ν differ at most at a countable subset of $[0,1]$. We claim that if a weak*-null sequence $(\nu_n)_{n\geq 1}$ in $M[0,1]$ admits a weak*-null extension $(\mu_n)_{n\geq 1}$ in $M(\mathrm{DA})$ then $(G_{\nu_n})_{n\geq 1}$ converges to zero pointwise outside a countable subset of $[0,1]$. Namely, observe that $[(0,0),(t,0)]$ is clopen and therefore (F_{μ_n}) converges to zero pointwise. Moreover, since $(\pi_1)_*\mu_n = \nu_n$, the maps F_{μ_n} and G_{ν_n} differ at most at a countable subset of $[0,1]$. This proves the claim.

In order to conclude the proof of the proposition, we simply exhibit a weak*-null sequence $(\nu_n)_{n\geq 1}$ in $M[0,1]$ such that $(G_{\nu_n})_{n\geq 1}$ does not converge to zero pointwise outside a countable subset of $[0,1]$. For instance, pick a sequence of intervals $[a_n, b_n] \subset [0,1]$ such that $\lim_{n\to+\infty}(b_n - a_n) = 0$ and set $\nu_n = \delta_{a_n} - \delta_{b_n}$; the uniform continuity of each $f \in C[0,1]$ implies that the sequence $(\nu_n)_{n\geq 1}$ is weak*-null. Note that G_{ν_n} equals the characteristic function $\chi_{[a_n,b_n[}$ and that the intervals can be chosen in a way that $(G_{\nu_n})_{n\geq 1}$ does not converge to zero at any point of $[0,1[$. □

Remark 2.10. It is well-known that the map

$$C(\mathrm{DA}) \ni f \longmapsto \big(f(t,1) - f(t,0)\big)_{t\in[0,1]} \in c_0[0,1]$$

induces an isomorphism between $C(\mathrm{DA})/\pi_1^* C[0,1]$ and $c_0[0,1]$. Thus, the quotient $C(\mathrm{DA})/\pi_1^* C[0,1]$ is WCG. In view of Proposition 2.9, we thus have an example of a $C(K)$ space and a separable Banach subalgebra $\phi^* C(L)$ such that $C(K)/\phi^* C(L)$ is WCG, but $\phi^* C(L)$ does not have the c_0EP in $C(K)$.

We finish the section by showing (Example 2.12) that there exists a continuous increasing surjection $\phi : K \to L$ such that K and L are countable compact lines and $\phi^* C(L)$ is not complemented in $C(K)$. To prove that $\phi^* C(L)$ is not complemented in $C(K)$ we will use [5, Lemma 2.7].

We recall some definitions. Let X be a linearly ordered set and A be a subset of X. A point $x \in X$ is a *right limit point* of A (relatively to X) if x is not the maximum of X and for every $y \in X$ with $y > x$ we have $]x, y[\cap A \neq \emptyset$. Similarly, one defines left limit points. We denote by $\mathrm{der}(A, X)$ the set of points $a \in A$ that are both right and left limits of A relatively to X. The points of $\mathrm{der}(X, X)$ are called *internal points* of X (in accordance with the terminology of [5]). Recursively, we define $\mathrm{der}_n(A, X)$ by setting $\mathrm{der}_0(A, X) = A$ and $\mathrm{der}_{n+1}(A, X) = \mathrm{der}\big(\mathrm{der}_n(A, X), X\big)$. The *internal order* of A (relatively to X) is defined by $\mathrm{io}(A, X) = n - 1$, where n is the least natural number with $\mathrm{der}_n(A, X) = \emptyset$; we set $\mathrm{io}(A, X) = +\infty$ if $\mathrm{der}_n(A, X) \neq \emptyset$, for all n.

In [5, Lemma 2.7, (2)] the authors prove that, given compact lines K, L and a continuous increasing surjection $\phi : K \to L$, the Banach subalgebra $\phi^* C(L)$ of $C(K)$ is complemented if and only if $\mathrm{io}(Q, L) < +\infty$, where Q is the set of points $p \in \mathrm{der}(L, L)$ such that $\phi^{-1}(p)$ has more than one point.

Remark 2.11. The following observation is useful for constructing compact lines. Given a compact line K and a family of nonempty compact lines $(L_x)_{x \in K}$, the set $\bigcup_{x \in K} \big(\{x\} \times L_x\big)$ endowed with the lexicographic order is also a compact line.

Example 2.12. Let $B = \big\{\pm\frac{1}{k} : k = 1, 2, \dots\big\} \cup \{0\}$ be endowed with the standard order of real numbers. For each positive integer n, denote by B_n the set of those $(x_1, \dots, x_n) \in B^n$ such that $x_i = 0$ implies $x_{i+1} = 0$, for all $i = 1, \dots, n-1$. Endow B_n with the lexicographic order. Using Remark 2.11, it follows easily by induction that B_n is a compact line, for all n. Moreover, for $i = 1, \dots, n-1$, we have $\mathrm{der}_i(B_n, B_n) = B_{n-i} \times \{0\}^i$ and thus $\mathrm{io}(B_n, B_n) = n$. Denote by L the disjoint union $\big(\bigcup_{n=1}^{\infty} B_n\big) \cup \{\infty\}$, ordered so that the elements of B_n precede the elements of B_{n+1} and such that ∞ is the largest element of L. Then L is a countable compact line. Since B_n is a convex subset of L, we have $\mathrm{io}(B_n, L) = \mathrm{io}(B_n, B_n) = n$ and therefore $\mathrm{io}(L, L) = +\infty$, since $\mathrm{io}(B_n, L) \leq \mathrm{io}(L, L)$, for all n. Let $K = L \times \{0, 1\}$ be endowed with the lexicographic order and $\phi : K \to L$ denote the first projection. Then K is also a countable compact line and ϕ is a continuous increasing surjection. The set Q is equal to $\mathrm{der}(L, L)$ and therefore $\mathrm{io}(Q, L) = +\infty$. It follows from [5, Lemma 2.7, (2)] that $\phi^* C(L)$ is not complemented in $C(K)$.

3. Main result

In this section we prove our main result, which is the following.

Theorem 3.1. *Let K be a compact line, L be a countable compact Hausdorff space and $\phi : K \to L$ be a continuous surjection. Then, for any $\varepsilon > 0$, the subspace $\phi^* C(L)$ has the $(2 + \varepsilon)$-c_0EP in $C(K)$.*

Recall that every nonempty countable compact Hausdorff space is homeomorphic to an ordinal segment $[0, \alpha]$, for some $\alpha < \omega_1$. We will prove

Theorem 3.1 using induction on α. The hard case is when α is a limit ordinal. The following result takes care of that.

Theorem 3.2. *Let K be a compact line, L be a compact Hausdorff space and $\phi : K \to L$ be a continuous surjection. Let $(L_\gamma)_{\gamma\in\Gamma}$ be a disjoint family of clopen subsets of L such that $L \setminus \big(\bigcup_{\gamma\in\Gamma} L_\gamma\big)$ has at most one point. Set $K_\gamma = \phi^{-1}[L_\gamma]$ and $\phi_\gamma = \phi|_{K_\gamma} : K_\gamma \to L_\gamma$. Let $\lambda \geq 2$ and assume that $(\phi_\gamma)^*C(L_\gamma)$ has the λ-c_0EP in $C(K_\gamma)$, for all $\gamma \in \Gamma$. Then, for any $\varepsilon > 0$, $\phi^*C(L)$ has the $(\lambda + \varepsilon)$-c_0EP in $C(K)$.*

Let us start by proving Theorem 3.1 using Theorem 3.2. The proof of Theorem 3.2 requires several technical lemmas and will be presented in Subsection 3.1 below.

Proof of Theorem 3.1. For every countable ordinal α, denote by $\mathbb{P}(\alpha)$ the condition which asserts that the thesis of the theorem holds for $L = [0, \alpha]$, any compact line K and any continuous surjection $\phi : K \to L$. If α is finite then $\phi^*C[0, \alpha]$ has the 1-c_0EP in $C(K)$ because a right inverse s of ϕ is continuous and $\phi^* \circ s^* : C(K) \to \phi^*C[0, \alpha]$ is a norm-1 projection. If α is infinite then $[0, \alpha]$ and $[0, \alpha + 1]$ are homeomorphic, so $\mathbb{P}(\alpha)$ is equivalent to $\mathbb{P}(\alpha + 1)$. To conclude the proof, let $\alpha < \omega_1$ be a limit ordinal and set $L = [0, \alpha]$. Let $(\alpha_n)_{n\geq 0}$ be a strictly increasing sequence of ordinals with $\sup_{n\geq 0} \alpha_n = \alpha$. Set $L_0 = [0, \alpha_0]$ and $L_n = \,]\alpha_{n-1}, \alpha_n]$, for $n \geq 1$. Each L_n is homeomorphic to $[0, \beta_n]$, for some $\beta_n < \alpha$; therefore, given $\varepsilon > 0$, the subspace $(\phi_n)^*C(L_n)$ has the $(2 + \varepsilon)$-c_0EP in $C(K_n)$, where $K_n = \phi^{-1}[L_n]$ and $\phi_n = \phi|_{K_n} : K_n \to L_n$. Hence, Theorem 3.2 yields that $\phi^*C(L)$ has the $(2 + 2\varepsilon)$-c_0EP in $C(K)$. □

Theorem 3.1 has some interesting corollaries concerning the complementation of isomorphic copies of c_0 in $C(K)$ spaces.

Corollary 3.3. *Let K be a compact line and $(f_n)_{n\geq 1}$ be a sequence in $C(K)$ equivalent to the canonical basis of c_0. If the map:*

$$\phi : K \ni p \longmapsto \big(f_n(p)\big)_{n\geq 1} \in \mathbb{R}^\omega \tag{2}$$

has countable range then the isomorphic copy of c_0 spanned by $(f_n)_{n\geq 1}$ is complemented in $C(K)$.

Proof. Note that the Banach subalgebra spanned by $(f_n)_{n\geq 1}$ is $\phi^*C(L)$, where L denotes the range of ϕ (endowed with the topology induced by the product topology of $\mathbb{R}^\omega$). □

Corollary 3.4. *Let K be a compact line and $(f_n)_{n\geq 1}$ be a sequence in $C(K)$ equivalent to the canonical basis of c_0. Assume that each f_n has countable range and that there exists $\varepsilon > 0$ such that, for all $n \geq 1$ and all $p \in K$, $|f_n(p)| \geq \varepsilon$ if $f_n(p) \neq 0$. Then the isomorphic copy of c_0 spanned by $(f_n)_{n\geq 1}$ is complemented in $C(K)$.*

Proof. Let $T \in \mathcal{B}\big(c_0, C(K)\big)$ be the map that carries the canonical basis of c_0 to $(f_n)_{n\geq 1}$. Note that $\sum_{n=1}^{\infty} |f_n(p)| = \|T^*\delta_p\| \leq \|T\|$, for all $p \in K$. Thus, our assumptions imply that, for each $p \in K$, the sequence $\big(f_n(p)\big)_{n\geq 1}$ is eventually null; since each f_n has countable range, it follows that the map (2) has countable range as well. □

The next proposition implies that the constant 2 in the thesis of Theorem 3.1 is optimal.

Proposition 3.5. *Let K be a Boolean space having more than one limit point. Then there exists a continuous surjection $\phi : K \to [0, \omega]$ such that $\phi^* C[0, \omega]$ does not have the λ-c_0EP in $C(K)$ for $\lambda < 2$.*

Proof. The space K can be written as a disjoint union $K = B_0 \cup B_1$, with B_0, B_1 infinite clopen subsets of K. For $i = 0, 1$, let $(B_i^k)_{k\geq 0}$ be a sequence of nonempty disjoint clopen subsets of B_i. Define the map ϕ by setting $\phi(p) = 2k + i$ for $p \in B_i^k$ and $\phi(p) = \omega$ otherwise. Consider the normalized weak*-null sequence $(v_n)_{n\geq 0}$ in $\ell_1[0, \omega] \equiv C[0, \omega]^*$ defined as follows: set $v_n(n) = \frac{1}{2}$, $v_n(n + 1) = -\frac{1}{2}$ and $v_n(j) = 0$ for $j \in [0, \omega] \setminus \{n, n + 1\}$. We claim that if $(\mu_n)_{n\geq 0}$ is a weak*-null sequence in $M(K)$ such that $\phi_*\mu_n = v_n$, for all $n \geq 0$, then $\sup_{n\geq 0} \|\mu_n\| \geq 2$. Note that the equality $\phi_*\mu_n = v_n$ means that $\mu_n(B_i^k) = v_n(2k + i)$ and that $\mu_n(K) = 0$. Set $A_i = \bigcup_{k=0}^{\infty} B_i^k$, so that $|\mu_n(A_i)| = \frac{1}{2}$. Now:

$$\begin{aligned}|\mu_n|(B_i) &\geq |\mu_n(A_i)| + |\mu_n(B_i \setminus A_i)| = |\mu_n(A_i)| + |\mu_n(B_i) - \mu_n(A_i)|\\ &\geq 2|\mu_n(A_i)| - |\mu_n(B_i)| = 1 - |\mu_n(B_i)|,\end{aligned}$$

and:

$$\|\mu_n\| = |\mu_n|(B_0) + |\mu_n|(B_1) \geq 2 - |\mu_n(B_0)| - |\mu_n(B_1)|.$$

Since B_i is clopen, $\lim_{n\to+\infty} \mu_n(B_i) = 0$ and the claim is proved. □

Remark 3.6. In Theorem 3.1, the assumption that L be countable can be weakened: namely, it is sufficient to assume that L be hereditarily paracompact and scattered. To see this, consider the class of all compact Hausdorff spaces L such that the thesis of Theorem 3.1 holds for any compact line K and any continuous surjection $\phi : K \to L$. Theorem 3.2 implies that such class is closed under the operation of taking the Alexandroff compactification of topological sums. Moreover, it is known [3, Theorem 3, (3)] that the smallest class of spaces closed under such operation and containing the singleton is the class of compact Hausdorff hereditarily paracompact scattered spaces.

3.1. **Some technical lemmas and the proof of Theorem 3.2.** Recall that if F is a closed subset of K then an *extension operator* for F is a bounded operator $E_F : C(F) \to C(K)$ which is a right inverse for the restriction operator $\rho_F : C(K) \to C(F)$. We call E_F a *regular* extension operator if, in addition, $\|E_F\| \leq 1$ and $E_F(\mathbf{1}_F) = \mathbf{1}_K$. If K is a compact line then every nonempty closed subset of K admits a regular extension operator ([6, Lemma 4.2]).

Lemma 3.7. *Let K be a compact Hausdorff space, F be a closed subset of K and $E_F : C(F) \to C(K)$ be a regular extension operator. Let $(p_\theta)_{\theta\in\Theta}$ be a family of distinct points of $K \setminus F$ such that every limit point of $\{p_\theta : \theta \in \Theta\}$ is in F. Then the equality:*

$$P(f)(\theta) = \big(f - E_F(f|_F)\big)(p_\theta), \quad f \in C(K),\ \theta \in \Theta, \tag{3}$$

defines a bounded linear map $P : C(K) \to c_0(\Theta)$. Moreover, for every $v \in c_0(\Theta)^* \equiv \ell_1(\Theta)$, the measure $P^*(v) \in M(K)$ satisfies:

(i) *$P^*(v)(K) = 0$;*
(ii) *$P^*(v)$ and $\sum_{\theta\in\Theta} v(\theta)\delta_{p_\theta}$ define the same Borel measure on $K \setminus F$;*
(iii) *the total variation $|P^*(v)|$ satisfies $|P^*(v)|(F) \le \|v\|$.*

Proof. Given $f \in C(K)$, since every limit point of $\{p_\theta : \theta \in \Theta\}$ is in the zero set of $f - E_F(f|_F)$, it follows that $P(f)$ indeed belongs to $c_0(\Theta)$. From $P(\mathbf{1}_K) = 0$ we obtain (i). It remains to prove (ii) and (iii). Define a bounded operator $P_1 : C(K) \to \ell_\infty(\Theta)$ by setting $P_1(f)(\theta) = f(p_\theta)$. Then P, regarded with counterdomain in $\ell_\infty(\Theta)$, can be written as $P = P_1 - P_2$, where $P_2 = P_1 \circ E_F \circ \rho_F$. Using the standard bilinear pairing between $\ell_1(\Theta)$ and $\ell_\infty(\Theta)$, we can regard $v \in \ell_1(\Theta)$ as linear functional on $\ell_\infty(\Theta)$. We can therefore write $P^*(v) = P_1^*(v) - P_2^*(v)$. Clearly, $P_1^*(v) = \sum_{\theta\in\Theta} v(\theta)\delta_{p_\theta}$. To prove (ii), we now check that $P_2^*(v)$ vanishes identically on $K \setminus F$. Namely, if $i : F \to K$ denotes the inclusion map then $\rho_F = i^*$ and thus the adjoint of ρ_F is the push-forward $i_* : M(F) \to M(K)$ that extends measures to zero. Finally, to prove (iii), note that $P_1^*(v)$ vanishes identically on F, and hence:

$$|P^*(v)|(F) = |P_2^*(v)|(F) \le \|P_2^*(v)\| \le \|v\|. \qquad \square$$

Lemma 3.8. *Let $(\mathfrak{X}, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to \mathbb{R}$ be a signed measure with $\mu(\mathfrak{X}) = 0$. If $f : \mathfrak{X} \to \mathbb{R}$ is a bounded measurable function then:*

$$\Big|\int_{\mathfrak{X}} f \,\mathrm{d}\mu\Big| \le \frac{1}{2}\,(\sup f - \inf f)\|\mu\|. \tag{4}$$

Proof. Since $\mu(\mathfrak{X}) = 0$, neither side of inequality (4) changes if we add a constant to f. Thus, we can assume $\sup f = -\inf f$, in which case $\sup_{x\in\mathfrak{X}} |f(x)| \le \frac{1}{2}(\sup f - \inf f)$. $\square$

Remark 3.9. Note that if $\mu \in M(K)$ and $(U_\theta)_{\theta\in\Theta}$ is a (possibly uncountable) family of disjoint open subsets of K then:

$$\mu(B) = \sum_{\theta\in\Theta} \mu(B \cap U_\theta),$$

for any Borel subset B of $U = \bigcup_{\theta\in\Theta} U_\theta$; moreover, if $f \in L^1(K, \mu)$ then:

$$\int_U f \,\mathrm{d}\mu = \sum_{\theta\in\Theta} \int_{U_\theta} f \,\mathrm{d}\mu.$$

To prove these equalities, note that $|\mu|(U_\theta) = 0$ for $\theta \in \Theta$ outside some countable subset Θ_0 of Θ and that, by regularity, μ vanishes identically on $\bigcup_{\theta\in\Theta\setminus\Theta_0} U_\theta$.

Lemma 3.10. *Let K be a compact line and $(I_\theta)_{\theta\in\Theta}$ be a disjoint family of clopen intervals of K such that $\bigcup_{\theta\in\Theta} I_\theta$ is a proper subset of K. For each $\theta \in \Theta$, let $(\nu_\theta^n)_{n\geq 1}$ be a weak*-null sequence in $M(I_\theta)$. Assume that:*

$$\sup_{n\geq 1}\sum_{\theta\in\Theta}\|\nu_\theta^n\| < +\infty. \tag{5}$$

Then, there exists a weak-null sequence $(\nu^n)_{n\geq 1}$ in $M(K)$ such that, for all $n \geq 1$:*

(a) $\nu^n|_{I_\theta} = \nu_\theta^n$, *for all* $\theta \in \Theta$;
(b) $\nu^n(K) = 0$;
(c) $\|\nu^n\| \leq \sum_{\theta\in\Theta}\big(\|\nu_\theta^n\| + |\nu_\theta^n(I_\theta)|\big)$.

Proof. Assuming without loss of generality that all I_θ are nonempty, pick $p_\theta \in I_\theta$, for each $\theta \in \Theta$. Set $F = K \setminus \bigcup_{\theta\in\Theta} I_\theta$ and note that every limit point of $\{p_\theta : \theta \in \Theta\}$ is in F. Since F is a nonempty closed subset of the compact line K, there exists a regular extension operator E_F. Define P as in (3). Our plan is the following: we will start by defining a certain weak*-null sequence $(v^n)_{n\geq 1}$ in $\ell_1(\Theta) \equiv c_0(\Theta)^*$; using P, this yields a weak*-null sequence of measures $P^*(v^n) \in M(K)$. Then, since $\sum_{\theta\in\Theta}\|\nu_\theta^n\| < +\infty$, it is easily seen that there exists a unique $\nu^n \in M(K)$ satisfying (a) and $\nu^n|_F = P^*(v^n)|_F$.

We define v^n by setting $v^n(\theta) = \nu_\theta^n(I_\theta)$. By (5), the sequence $(v^n)_{n\geq 1}$ is bounded in $\ell_1(\Theta)$ and, since $(\nu_\theta^n)_{n\geq 1}$ is weak*-null, we have that $v^n(\theta) \to 0$ for each θ. Thus, $(v^n)_{n\geq 1}$ is weak*-null. As explained above, from v^n we obtain $\nu^n \in M(K)$. Using (ii) of Lemma 3.7, we obtain:

$$P^*(v^n)(K \setminus F) = \sum_{\theta\in\Theta} v^n(\theta) = \nu^n(K \setminus F)$$

and therefore $\nu^n(K) = P^*(v^n)(K) = 0$, by (i) of Lemma 3.7. This proves (b). To prove (c), note that:

$$\|\nu^n\| = |P^*(v^n)|(F) + \sum_{\theta\in\Theta}\|\nu_\theta^n\|,$$

and use (iii) of Lemma 3.7.

Finally, we prove that $(\nu^n)_{n\geq 1}$ is weak*-null. Since $(\nu^n)_{n\geq 1}$ is bounded and the increasing functions form a linearly dense subset of $C(K)$ ([6, Proposition 3.2]), it suffices to show that $\int_K f\,\mathrm{d}\nu^n \to 0$, for any continuous increasing function $f : K \to \mathbb{R}$. Recalling that $\big(P^*(v^n)\big)_{n\geq 1}$ is weak*-null, it suffices to show that:

$$\lim_{n\to+\infty}\int_K f\,\mathrm{d}\big(\nu^n - P^*(v^n)\big) = 0.$$

The measure $\nu^n - P^*(v^n)$ vanishes identically on F and, by (ii) of Lemma 3.7, $P^*(v^n)$ equals $v^n(\theta)\delta_{p_\theta}$ on I_θ; therefore:

$$\int_K f \,\mathrm{d}\big(\nu^n - P^*(v^n)\big) = \sum_{\theta\in\Theta} \int_{I_\theta} f \,\mathrm{d}\big(\nu^n_\theta - v^n(\theta)\delta_{p_\theta}\big). \tag{6}$$

Since $(\nu^n_\theta)_{n\ge1}$ is weak*-null and $v^n(\theta) \to 0$, we have that each term of the sum on the righthand side of (6) tends to zero. To conclude the proof, we will show that, given $\varepsilon > 0$, there exists a finite subset $\Phi \subset \Theta$ such that:

$$\Big| \sum_{\theta\in\Theta\setminus\Phi} \int_{I_\theta} f \,\mathrm{d}\big(\nu^n_\theta - v^n(\theta)\delta_{p_\theta}\big)\Big| < \varepsilon,$$

for all $n \ge 1$. Write $I_\theta = [a_\theta, b_\theta]$ and $K = [a, b]$. Using that f is increasing, we obtain:

$$\sum_{\theta\in\Theta} \big(f(b_\theta) - f(a_\theta)\big) \le f(b) - f(a) < +\infty;$$

in particular, there exists a finite subset Φ of Θ such that $f(b_\theta) - f(a_\theta) \le \varepsilon$, for $\theta \in \Theta \setminus \Phi$. Noting that $(\nu^n_\theta - v^n(\theta)\delta_{p_\theta})(I_\theta) = 0$, Lemma 3.8 yields:

$$\begin{aligned}\Big| \int_{I_\theta} f \,\mathrm{d}\big(\nu^n_\theta - v^n(\theta)\delta_{p_\theta}\big)\Big| &\le \frac{1}{2}\big(f(b_\theta) - f(a_\theta)\big)\|\nu^n_\theta - v^n(\theta)\delta_{p_\theta}\| \\ &\le \big(f(b_\theta) - f(a_\theta)\big)\|\nu^n_\theta\|.\end{aligned}$$

Hence:

$$\Big| \sum_{\theta\in\Theta\setminus\Phi} \int_{I_\theta} f \,\mathrm{d}\big(\nu^n_\theta - v^n(\theta)\delta_{p_\theta}\big)\Big| \le \sum_{\theta\in\Theta\setminus\Phi} \big(f(b_\theta) - f(a_\theta)\big)\|\nu^n_\theta\| \le \varepsilon \sum_{\theta\in\Theta} \|\nu^n_\theta\|.$$

Having (5) in mind, the conclusion follows. □

Lemma 3.11. *Let $(r^n_\gamma)_{n\ge1,\gamma\in\Gamma}$ be a family of real numbers, with Γ a countable set. Assume that $\lim_{n\to+\infty} r^n_\gamma = 0$, for all $\gamma \in \Gamma$. Then, there exists an increasing sequence $(\Phi_n)_{n\ge1}$ of finite subsets of Γ such that $\Gamma = \bigcup_{n=1}^\infty \Phi_n$ and $\lim_{n\to+\infty} \sum_{\gamma\in\Phi_n} r^n_\gamma = 0$.*

Proof. If Γ is finite, simply take $\Phi_n = \Gamma$, for all $n \ge 1$. If Γ is infinite, we can obviously assume $\Gamma = \omega$. For each $k \ge 1$, since $\lim_{n\to+\infty} \sum_{i=0}^k r^n_i = 0$, we can find $N_k \ge 1$ such that $\big|\sum_{i=0}^k r^n_i\big| < \frac{1}{k}$ for $n \ge N_k$. Moreover, we can assume that the sequence $(N_k)_{k\ge1}$ is strictly increasing. Now, for each $n \ge 1$, let $\varphi(n)$ be the largest positive integer k such that $N_k \le n$ (set $\varphi(n) = 0$, if there is no such k). Clearly $\varphi(n)$ increases with n and, since $\varphi(N_k) \ge k$, we have $\lim_{n\to+\infty} \varphi(n) = +\infty$. Thus, setting $\Phi_n = \big\{i \in \omega : i \le \varphi(n)\big\}$, we obtain that $(\Phi_n)_{n\ge1}$ is increasing and $\bigcup_{n=1}^\infty \Phi_n = \omega$. Finally, given $n \ge 1$, if $\varphi(n) \ne 0$ then $N_{\varphi(n)} \le n$ and therefore:

$$\Big|\sum_{i\in\Phi_n} r^n_i\Big| = \Big|\sum_{i=0}^{\varphi(n)} r^n_i\Big| < \frac{1}{\varphi(n)},$$

proving that $\lim_{n\to+\infty} \sum_{i\in\Phi_n} r^n_i = 0$. □

Finally, we can prove Theorem 3.2.

Proof of Theorem 3.2. By Corollary 2.4, it suffices to consider a normalized weak*-null sequence $(\mu^n)_{n\geq 1}$ in $M(L)$ and obtain a weak*-null sequence $(\tilde{\mu}^n)_{n\geq 1}$ in $M(K)$ such that $\phi_*\tilde{\mu}^n = \mu^n$, for all $n \geq 1$, and:

$$\limsup_{n\to+\infty} \|\tilde{\mu}^n\| \leq \lambda.$$

Set $\mu^n_\gamma = \mu^n|_{L_\gamma}$; since L_γ is clopen in L, the sequence $(\mu^n_\gamma)_{n\geq 1}$ is weak*-null in $M(L_\gamma)$. Using the assumption that $(\phi_\gamma)^*C(L_\gamma)$ has the λ-c_0EP in $C(K_\gamma)$ and using Lemma 2.3, we obtain a weak*-null sequence $(\hat{\mu}^n_\gamma)_{n\geq 1}$ in $M(K_\gamma)$ such that $(\phi_\gamma)_*\hat{\mu}^n_\gamma = \mu^n_\gamma$ and $\|\hat{\mu}^n_\gamma\| \leq \lambda\|\mu^n_\gamma\|$, for all $n \geq 1$.

Let us first handle the trivial case in which $L = \bigcup_{\gamma\in\Gamma} L_\gamma$. In this case, $K = \bigcup_{\gamma\in\Gamma} K_\gamma$ defines a finite partition of K and we obtain the measure $\tilde{\mu}^n \in M(K)$ by setting $\tilde{\mu}^n|_{K_\gamma} = \hat{\mu}^n_\gamma$, for all $\gamma \in \Gamma$. Now assume that $L \setminus \bigcup_{\gamma\in\Gamma} L_\gamma$ contains a single point, which we denote by ∞. Let Γ_0 be a countable subset of Γ such that $\mu^n_\gamma = 0$ for all $n \geq 1$ and $\gamma \in \Gamma \setminus \Gamma_0$.

Lemma 3.10 requires a family $(I_\theta)_{\theta\in\Theta}$ of disjoint clopen intervals. Thus, for each $\gamma \in \Gamma$, we write the clopen subset K_γ of K as a finite disjoint union $K_\gamma = \bigcup_{j=1}^{m_\gamma} I_{\gamma,j}$ of clopen intervals $I_{\gamma,j}$ of K. Set:

$$\Theta = \big\{(\gamma, j) : \gamma \in \Gamma,\ j = 1, \dots, m_\gamma\big\}.$$

Note that $\bigcup_{\theta\in\Theta} I_\theta = \bigcup_{\gamma\in\Gamma} K_\gamma$ is a proper subset of K, its complement being $\phi^{-1}(\infty)$. We now need a weak*-null sequence $(\nu^n_\theta)_{n\geq 1}$ in $M(I_\theta)$. We start by defining a weak*-null sequence $(\tilde{\mu}^n_\gamma)_{n\geq 1}$ in $M(K_\gamma)$ and then we set $\nu^n_{\gamma,j} = (\tilde{\mu}^n_\gamma)|_{I_{\gamma,j}}$.

For $\gamma \in \Gamma_0$, set $r^n_\gamma = \sum_{j=1}^{m_\gamma} |\hat{\mu}^n_\gamma(I_{\gamma,j})|$. Since $(\hat{\mu}^n_\gamma)_{n\geq 1}$ is weak*-null in $M(K_\gamma)$ and $I_{\gamma,j}$ is clopen in K_γ, we have that $\lim_{n\to+\infty} r^n_\gamma = 0$. Applying Lemma 3.11 to the family $(r^n_\gamma)_{n\geq 1,\gamma\in\Gamma_0}$, we obtain an increasing sequence $(\Phi_n)_{n\geq 1}$ of finite subsets of Γ_0. For $\gamma \in \Gamma$ and $n \geq 1$, we define $\tilde{\mu}^n_\gamma \in M(K_\gamma)$ as follows. If $\gamma \in \Phi_n$, we set $\tilde{\mu}^n_\gamma = \hat{\mu}^n_\gamma$; if $\gamma \in \Gamma \setminus \Phi_n$, we take $\tilde{\mu}^n_\gamma$ to be a Hahn–Banach extension of μ^n_γ. Note that, in any case, we have:

$$(\phi_\gamma)_*\tilde{\mu}^n_\gamma = \mu^n_\gamma \tag{7}$$

and $\|\tilde{\mu}^n_\gamma\| \leq \lambda\|\mu^n_\gamma\|$. More specifically:

$$\|\tilde{\mu}^n_\gamma\| \leq \lambda\|\mu^n_\gamma\|,\ \text{for } \gamma \in \Phi_n, \quad \|\tilde{\mu}^n_\gamma\| = \|\mu^n_\gamma\|,\ \text{for } \gamma \in \Gamma \setminus \Phi_n. \tag{8}$$

Note also that $(\tilde{\mu}^n_\gamma)_{n\geq 1}$ is indeed a weak*-null sequence in $M(K_\gamma)$, for all $\gamma \in \Gamma$; namely, the sequence is zero for $\gamma \notin \Gamma_0$ and, for $\gamma \in \Gamma_0$, we have $\tilde{\mu}^n_\gamma = \hat{\mu}^n_\gamma$, for n sufficiently large.

We are now ready to apply Lemma 3.10. Observe that:

$$\sum_{\theta\in\Theta} \|\nu^n_\theta\| = \sum_{\gamma\in\Gamma} \|\tilde{\mu}^n_\gamma\| \leq \lambda \sum_{\gamma\in\Gamma} \|\mu^n_\gamma\| \leq \lambda\|\mu^n\| = \lambda,$$

so that assumption (5) holds. Let $(\nu^n)_{n\ge 1}$ be the weak*-null sequence in $M(K)$ given by the lemma. It follows from (a) that:

$$\nu^n|_{K_\gamma} = \tilde{\mu}^n_\gamma. \tag{9}$$

Moreover, it follows from (c) that:

$$\|\nu^n\| \le \sum_{\gamma\in\Gamma} \|\tilde{\mu}^n_\gamma\| + \sum_{(\gamma,j)\in\Theta} |\tilde{\mu}^n_\gamma(I_{\gamma,j})|. \tag{10}$$

We claim that $\limsup_{n\to+\infty} \|\nu^n\| \le \lambda$. Using (8) we obtain:

$$\sum_{\gamma\in\Gamma} \|\tilde{\mu}^n_\gamma\| \le \lambda \sum_{\gamma\in\Phi_n} \|\mu^n_\gamma\| + \sum_{\gamma\in\Gamma\setminus\Phi_n} \|\mu^n_\gamma\|, \tag{11}$$

$$\sum_{(\gamma,j)\in\Theta} |\tilde{\mu}^n_\gamma(I_{\gamma,j})| \le \sum_{\gamma\in\Phi_n} r^n_\gamma + \sum_{\gamma\in\Gamma\setminus\Phi_n} \|\mu^n_\gamma\|. \tag{12}$$

From (10), (11), (12), and the fact that $\lambda \ge 2$, it follows that:

$$\|\nu^n\| \le \lambda \sum_{\gamma\in\Gamma} \|\mu^n_\gamma\| + \sum_{\gamma\in\Phi_n} r^n_\gamma \le \lambda + \sum_{\gamma\in\Phi_n} r^n_\gamma.$$

Since $\lim_{n\to+\infty} \sum_{\gamma\in\Phi_n} r^n_\gamma = 0$, the claim is proved.

Finally, pick $p \in \phi^{-1}(\infty)$ and set $\tilde{\mu}^n = \nu^n + \mu^n(L)\delta_p$. As $\mu^n(L) \to 0$, we have that $\limsup_{n\to+\infty} \|\tilde{\mu}^n\| \le \lambda$ and that $(\tilde{\mu}^n)_{n\ge 1}$ is weak*-null. It remains to check that $\phi_*\tilde{\mu}^n = \mu^n$, for all $n \ge 1$. Note that, for all $\gamma \in \Gamma$:

$$(\phi_*\tilde{\mu}^n)|_{L_\gamma} = (\phi_*\nu^n)|_{L_\gamma} = (\phi_\gamma)_*(\nu^n|_{K_\gamma}) \overset{(9)}{=} (\phi_\gamma)_*(\tilde{\mu}^n_\gamma) \overset{(7)}{=} \mu^n_\gamma = \mu^n|_{L_\gamma}.$$

Thus $\phi_*\tilde{\mu}^n$ and μ^n agree on all Borel subsets of $\bigcup_{\gamma\in\Gamma} L_\gamma$. Since the complement of $\bigcup_{\gamma\in\Gamma} L_\gamma$ in L has just one point, the proof will be concluded if we check that $(\phi_*\tilde{\mu}^n)(L) = \mu^n(L)$. By (b) of Lemma 3.10, $\nu^n(K) = 0$ and hence:

$$(\phi_*\tilde{\mu}^n)(L) = \tilde{\mu}^n(K) = \mu^n(L). \qquad \square$$

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