Blind Equalization Based on a Cascade of Optimal Two-Tap CM Equalizers

Denis Gustavo Fantinato[1,2], Romis Attux[1,2](Supervisor), Celso de Sousa Jr.[1], Ricardo Suyama[3], Aline Neves[3](Co-Supervisor), João Marcos Travassos Romano[1,4]

[1]Laboratory of Signal Processing for Communications (DSPCOM) – FEEC/UNICAMP
[2]Department of Computer Engineering and Industrial Automation (DCA) – FEEC/UNICAMP
[3]Engineering, Modeling and Applied Social Science Center (CECS) – UFABC
[4]Department of Microwave and Optics (DMO) – FEEC/UNICAMP

{denisgf, attux}@dca.fee.unicamp.br, celso_de_sousa_junior@yahoo.com.br,
{ricardo.suyama, aline.neves}@ufabc.edu.br, romano@dmo.fee.unicamp.br

Abstract – This article introduces a blind equalization methodology based on a cascade of two-tap finite impulse response filters globally optimal with respect to the constant modulus (CM) criterion. It is shown that a formulation of the CM cost function in terms of a mean squared error (MSE) metric and of Volterra-like constraints allow the optimization process to be carried out, when two parameters are to be adapted, with the aid of a one-dimensional search process that suits quite well an exhaustive framework. The proposal will be compared to a similar cascade of two-tap Wiener solutions and also with conventional Wiener filters.

Keywords – Unsupervised filtering, blind equalization, constant modulus algorithm (CMA).

This work is dedicated to those who have been part of the first 15 years of existence of the Laboratory of Signal Processing for Communications (DSPCOM).

"Vous avez confirmé dans ces lieux pleins d'ennui
Ce que Newton connut sans sortir de chez lui"
Voltaire

1. Introduction

The constant modulus algorithm (CMA) - the origin of which is associated with the classical works of [1] and [2] – is arguably the most thoroughly analyzed member of the class of Bussgang algorithms. This interest can be justified in theoretical terms – in view, for instance, of points of contact with kurtosis-based unsupervised methods [3] – and also from the standpoint of practical applicability.

The CMA is a stochastic version of the steepest descent method when the latter is applied to the task of minimizing the following cost function [4]:

$$J_{CM}(\mathbf{w}) = E[(|y(n)|^2 - R_2)^2] \qquad (1)$$

where $E[.]$ denotes statistical expectation, $R_2 = E[|s(n)|^4]/E[|s(n)|^2]$, $y(n)$ is the equalizer output and $s(n)$ is the transmitted sequence. Assuming that the equalizer is a finite impulse response (FIR) filter, the algorithm update expression is:

$$\mathbf{w}(n+1) = \mathbf{w}(n) - \mu\left(|y(n)|^2 - R_2\right)y(n)\mathbf{x}^*(n) \qquad (2)$$

where μ is the step-size and * denotes complex conjugation.

The CM cost function typically possesses global and local minima, as well as saddle points and a local maximum [4][5]. Therefore, depending on the chosen initial condition, the CMA may converge towards local minima, which means that the potential performance achievable by the filtering structure will not be actualized.

In this work, we propose a method to constructively build a blind equalizer as a cascade of two-tap filters that are globally optimized using a one-dimensional line search procedure in the context of a formulation of the CM cost function proposed in [6]. The idea of the method is to present an unsupervised design framework for which there would be a guarantee of optimality with respect to the obtained solution without the need for resorting to multidimensional exhaustive search. The method will be compared with an analogous cascade of two-tap equalizers and a conventional structure optimized in accordance with the Wiener criterion.

2. Formulation of the CM Criterion Using Quadratic Constraints

Consider that the transmitted signal is composed of binary (+1 / -1) samples and that all systems to be dealt with are composed exclusively of real-valued parameters. In this case, if a two-tap equalizer is employed, it is possible to rewrite the term $|y(n)|^2$ in (1) as:

$$|y(n)|^2 = y^2(n) = w_0^2 x^2(n) + 2w_0 w_1 x(n)x(n-1) + w_1^2 x^2(n-1) \quad (3)$$

This can be seen as the output of a Volterra filter with input vector $\boldsymbol{\xi}(n) = [x^2(n)\ x(n)x(n-1)\ x^2(n-1)]^T$ and parameter vector $\boldsymbol{\theta} = [\theta_0\ \theta_1\ \theta_2]^T = [w_0^2\ 2w_0w_1\ w_1^2]^T$. If this is the case, one may reinterpret (1) as a mean squared error (MSE) criterion with a constant reference signal: $R_2 = 1$. This is the starting point of a formulation of the CM criterion in terms of an MSE cost function subject to Volterra-like constraints. This idea is outlined in [7], but took a definite shape in [6]. Later, works like [8] and [9] analyzed the effects of relaxing the aforementioned constraints, which led to a lower bound for the CM cost function.

To summarize this discussion, for the two-tap case, the CM criterion can be written as:

$$\begin{gathered} \min_{\boldsymbol{\theta}} J_{MSE}(\boldsymbol{\theta}) = \min_{\boldsymbol{\theta}} E[(\upsilon(n) - R_2)^2] \\ s.t.\quad \theta_1^2 = 4\theta_0\theta_2 \end{gathered} \quad (4)$$

where $\upsilon(n) = \boldsymbol{\theta}^T\boldsymbol{\xi}(n)$. Using the method of Lagrange multipliers, the optimal solution is shown to be that satisfying:

$$\begin{gathered} \nabla J_{MSE}(\boldsymbol{\theta}) = \lambda \nabla J_{CST}(\boldsymbol{\theta}) \\ J_{CST}(\boldsymbol{\theta}) = 0 \end{gathered} \quad (5)$$

where $J_{CST}(\boldsymbol{\theta}) = \theta_1^2 - 4\theta_0\theta_2$ and λ is the Lagrange multiplier. Equation (4) and the essential results of Wiener filtering theory [10] show that the optimal parameter vector will have the form:

$$\boldsymbol{\theta} = [\mathbf{R} - \lambda\mathbf{C}]^{-1}\boldsymbol{p} \quad (6)$$

being $\mathbf{R} = E[\boldsymbol{\xi}(n)\boldsymbol{\xi}^T(n)]$, $\boldsymbol{p} = R_2\mathrm{E}[\boldsymbol{\xi}(n)]$ and

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \quad (7)$$

Although it is possible to use this solution and the constraint to find a polynomial expression for the possible values of λ, this rather laborious course will not be followed here. Rather, a line search procedure will be used to obtain the values of λ that allow the condition $J_{CST}(\boldsymbol{\theta}) = 0$ to hold. This is interesting, as the two-dimensional task of finding the global minimum of the CM cost function becomes a one-dimensional search problem, which can be solved by brute force methods in a more economical way. The analysis of the global character of the obtained solutions (which will include minima and saddle points) will be done by directly estimating the value of $J_{MSE}(\boldsymbol{\theta})$ using the received data $x(n)$.

The possibility of ensuring that global minima be obtained with a relatively parsimonious optimization method insinuates the possibility of building an equalizer as a cascade of two-tap equalizers that correspond to global CM optimal. The validity of this idea will be analyzed in the following, based on comparisons with Wiener solutions in both canonical and cascade forms.

3. Results

In order to find the global minima of the CM cost function, we must proceed, basically, according to three steps: first, it is necessary to estimate the correlation matrix $\mathbf{R}$ and the cross-correlation vector $\boldsymbol{p}$ in the Volterra domain; the second step corresponds to finding the desired value of λ, and the last step amounts to obtaining the constrained solution that leads to the optimal choice for $\mathbf{w}$ in terms of the CM cost.

The above defined first step is straightforward, since $\mathbf{R}$ and $\boldsymbol{p}$ can be easily estimated using a sample mean. A more elaborate discussion is demanded by the procedure concerning the solution for the Lagrange multiplier λ. The approach followed in this work is based in a linear search over λ, in which $\boldsymbol{\theta}$ – in Eq. (6) – and its correspondent $J_{CST}(\boldsymbol{\theta})$ are calculated for each value of λ belonging to a chosen limited set of discrete values sorted in ascending order. Theoretically, there exist exactly four real values of λ that satisfy $J_{CST}(\boldsymbol{\theta}) = 0$ [4][6], however, the exhaustive linear search assumes only certain discrete values, which will probably cause $J_{CST}(\boldsymbol{\theta})$ never to be strictly identical to zero. Nonetheless, a sufficiently small threshold value can be defined. In view of this, the search for the desired λ is feasible by seeking the minima of the cost function $J_{CST}^2(\boldsymbol{\theta})$. For a better visualization, we turn this problem into that of finding the maximum points in $f(J_{CST}(\boldsymbol{\theta}))$, in which $f(k) = 1/(1+k^2)$. Fig. 1 shows the obtained $f(J_{CST}(\boldsymbol{\theta}))$ in function of λ, considering the BPSK signal transmitted over the channel with impulse response $h_1(n) = 1 + 0.6z^{-1}$.

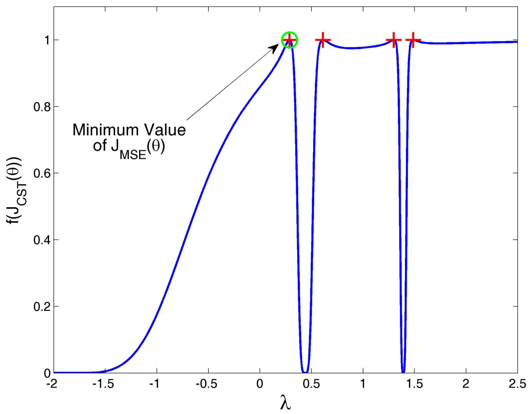

Figure 1: $f(J_{CST}(\boldsymbol{\theta}))$ and its maximum points.

The maxima of $f(J_{CST}(\boldsymbol{\theta}))$ are obtained with a peak detector. The peaks are displayed as red plus signs in Fig. 1. Among these, the suitable parameter λ is the one associated with the minimum value of $J_{MSE}(\boldsymbol{\theta})$. Finally, the conversion from the Volterra domain $\boldsymbol{\theta}$ to the filter domain $\mathbf{w}$ can be expressed as $w_0 = \sqrt{\theta_0}$, $w_1 = sign(\theta_1)\sqrt{\theta_2}$.

Interestingly, the same procedure can be iteratively performed to determine the next two-tap filter of a cascade; however, the input *x(n)* will be the output signal of the previous filter. Rigorously, **R** and ***p*** do not need to be determined each iteration, since the knowledge of **w** allows an analytical update. However, we will not follow this approach in this work.

From a communication system standpoint, the effect of a cascade can be interpreted as that of a single filter whose coefficients are given by the convolution of the cascade elements. This result is important, since, mathematically, a sufficient larger filter can be obtained by the combination of smaller ones. With this concept in mind, we proceed in analysis of the performance obtained by the cascaded filters accordingly to the presented approach. Firstly, we consider a maximum-phase channel $h_2(z) = 1 + 1.5z^{-1}$ and two cascaded FIR equalizers. We also analyze the results by comparing with a similar cascade of two-tap Wiener solutions – considering always the optimal source delay – and also with conventional Wiener filters. The Lagrange multiplier λ was varied from -10 to 10 in steps of 0.001. The resulting filter coefficients **w** for the two cascaded filters according to the global CM solution and Wiener solution are presented in Tab. 1.

	Two-Tap CM		Two-Tap Wiener	
	Filter 1	Filter 2	Filter 1	Filter 2
w_0	0.2480	0.1326	-0.2707	-0.1269
w_1	-0.5214	-1.0219	0.5865	1.0159

Table 1: Resulting coefficients for two-tap filters.

The results are interestingly quite similar for both filters, except for the fact that the signs of w_0 and w_1 are exchanged, which is expected in view of the unsupervised character of the CM cost function. The concatenated effect of both filters is represented as the convolution between them, which results in the coefficients shown in Tab. 2.

	2-tap CM Convolution	2-tap Wiener Convolution	Conventional 3-tap Wiener Solution
w_0	0.0329	0.0344	0.3259
w_1	-0.3226	-0.3494	-0.0517
w_2	0.5328	0.5958	1.0981

Table 2: Combined effect of the 2-tap filters and the conventional 3-tap Wiener solution.

In spite of their remaining close to each other, both two-tap filter solutions significantly differ from the conventional three-tap Wiener solution. To illustrate this point, we compare their performance in terms of MSE - always considering the optimal signal delay. Fig. 2 illustrates this case. It can be seen that the addition of elements to the cascade causes an MSE reduction; however, the equivalent conventional Wiener filter solution shows a more expressive improvement.

The behavior of the cascaded filters can show itself more clarifying if we consider a higher number of filters. With that in mind, we analyze the channel $h_3(z) = 0.1358 – 0.5566z^{-1} + 0.7412z^{-2} - 0.3470z^{-3} + 0.0440z^{-4}$ and 9 cascaded two-tap filters – equivalent up to a conventional 10-tap filter. The resulting performances in terms of MSE are shown in Fig. 3. In this case, the filter cascading effect becomes increasingly small and its associated MSE converges to a certain level, while the conventional Wiener filter shows clear signs of being capable of reaching smaller MSE levels.

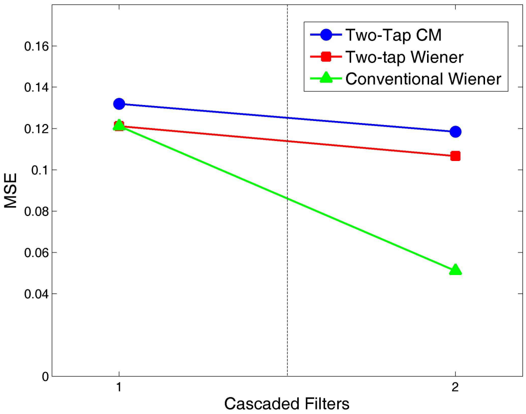

Figure 2: Performance of cascaded filters and equivalent conventional Wiener in terms of MSE for $h_2(n)$.

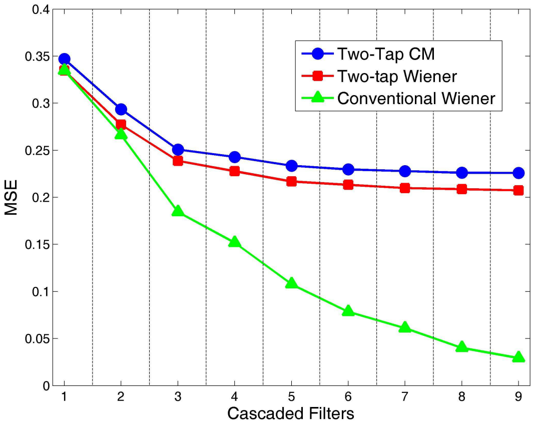

Figure 3: Performance of cascaded filters and equivalent conventional Wiener in terms of MSE for $h_3(n)$.

Although a more profound investigation is important, this saturation is probably due to the fact that the elements of the cascade, in view of their limited amount of degrees of freedom, cannot properly handle the process of optimal delay placement that the best Wiener solution incorporates in a natural way.

4. Conclusions

In this work, we proposed a blind methodology that reduces the problem of determining globally optimal two-tap FIR filters in terms of CM criterion to a one-dimensional search over a parameter λ. Moreover, we show that the task of blind equalization can be performed in terms of a cascade of two-tap equalizers. The results show that, the cascade is effective up to a certain level of MSE, being less effective than a “monolithic” Wiener filter in the long run. A deeper analysis, however, is required before more detailed conclusions be drawn.

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