Lower Bound of the Constant Modulus Criterion for Multilevel Modulations

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Abstract – The recently proposed lower bound of the classical blind CM criterion was shown to be able to work as an excellent blind equalizability index, which is a practical performance assessment metric in the context of inverse problems. However, there still remains the need of complementary studies aiming to cover a wider range of circumstances. Based on this, we present in this work an extension of the analysis of the lower bound by considering scenarios with multilevel modulated signals. For an 8-PAM and a 16-QAM signal, the simulations revealed the presence of a residual error offset, but even so, the validity of the index was preserved.

Keywords: blind equalization, constant modulus criterion, equalizability index, multilevel modulation.

This work is dedicated to those who have been part of the first 15 years of existence of the Laboratory of Signal Processing for Communications (DSPCOM).

1. Introduction

In digital communications systems, there exist certain relevant applications that demands real-time processing of a considerable amount of data, among which we can cite, for example, modern broadcasting services, like high-definition television (HDTV). In such tasks, there is an ever increasing challenge for larger data rates, implying in some requirements for the receiver to deal with. It is a common requirement, for instance, that the employed adaptive equalizers operate for baseband complex-valued data with multilevel modulated signals, such as 16-QAM, 64-QAM or even 128-QAM. Moreover, there may be no training sequence, causing the signal processing task to be performed in a blind mode.

A classical and well-established strategy used to deal with the problem of blind equalization is that based on the constant modulus (CM) criterion [1], which, along with its associated algorithm, the CMA, was an object of intense study for the last three decades [2]. Particularly, in view of the connections to Wiener theory [2] – a supervised formulation based on a mean-squared error (MSE) measure –, it was possible to introduce a polynomial formulation of the CM criterion [3], which brought a novel perspective to the use of fourth-order statistics in channel inversion. An interesting result derived from this approach was a lower bound for the CM cost function, which works as a measure for the attainable performance in channel deconvolution or, in other words, as a blind equalizability index. The original proposal was developed in [4] and later extended to the idiosyncrasies of the complex domain in [5], which allowed its application to a wider range of scenarios. However, as we intend to show in this work, another relevant step can be made if we consider that the transmitted signal is modulated according to constellations with relatively high order, like 8-PAM, 16-QAM or 64-QAM. If the validity of the lower bound is proved for such cases, in addition to its potential as an analytical tool, this measure – that reflects the residual MSE level for the supervised case – can be successfully used as practical performance assessment index in aid, for example, of equalization issues and of soft-demapping schemes in the HDTV receivers.

2. Polynomial Formulation of the CM Criterion

In order to recapitulate the concepts underlying the definition of the lower bound, we will present a brief derivation of the polynomial formulation of the CM criterion in the following. For further clarification, we recommend the reading of [5].

In its classical form, the CM cost function can be expressed as the minimization of the following cost function:

$$J_{CM}(\mathbf{w}) = E\left[\left(|y(n)|^2 - R_2\right)^2\right], \qquad (1)$$

where $R_2 = E\left[|s(n)|^4\right]/E\left[|s(n)|^2\right]$, $E[\cdot]$ denotes statistical expectation, $\mathbf{w}$ is the parameter vector of a finite impulse response (FIR) equalizer with N coefficients, $s(n)$ is the transmitted signal and $y(n)$ is the equalizer output signal. More specifically, the equalizer output can be defined as $y(n) = \mathbf{w}^H\mathbf{x}(n)$, in which $\mathbf{x}(n) = [x(n), x(n-1), \ldots, x(n-N+1)]$ is the equalizer input vector and $(\cdot)^H$ denotes Hermitian transposition.

Considering now the modified output signal defined as $\upsilon_c(n) = |y(n)|^2$, the polynomial formulation will take shape if we expand the second member, so that $|y(n)|^2 = y(n)y^*(n)$, where $(\cdot)^*$ denotes complex conjugation. For a better understanding, we assume, for instance, a two-tap complex equalizer, which prompt us to express

$$\begin{aligned}\upsilon_c(n) &= \begin{bmatrix} |w_0|^2 & w_0^* w_1 & w_0 w_1^* & |w_1|^2 \end{bmatrix} \\ &\quad\times \begin{bmatrix} |x(n)|^2 \\ x(n)x^*(n-1) \\ x^*(n)x(n-1) \\ |x(n-1)|^2 \end{bmatrix} \\ &= \boldsymbol{\theta}_c^H \boldsymbol{\xi}(n)\ ,\end{aligned} \tag{2}$$

where $\boldsymbol{\theta}_c$ is the polynomial filter parameter vector - which depends on $\mathbf{w}$ - and $\boldsymbol{\xi}(n)$ is the input vector in the Volterra domain [6]. In a similar manner, Eq. (2) could be extended to a generic N-tap complex equalizer.

By supposing $R_2 = 1$ and using the modified output signal $\upsilon_c(n)$ in Eq. (1), we have

$$J_{CM}(\boldsymbol{\theta}_c) = E\left[|\upsilon_c(n) - 1|^2\right], \tag{3}$$

which is the polynomial formulation of the CM cost.

Allowing now a modification with respect to the original CM cost, the next step is to admit a relaxation of the constraints regarding the influence of the values of $\mathbf{w}$ on $\boldsymbol{\theta}_c$ and make the polynomial filter parameters completely "free". Thus, the new unconstrained polynomial filter, hereinafter denoted simply as $\boldsymbol{\theta}$, will have as output the signal $\upsilon(n) = \boldsymbol{\theta}^H\boldsymbol{\xi}(n)$. Moreover, if we consider the autocorrelation matrix $\mathbf{R}_\xi$ and the cross-correlation vector $\mathbf{p}_\xi$ for $d(n) = R_2 = 1$ – in the Volterra domain – to be defined as

$$\mathbf{R}_\xi = E\left[\boldsymbol{\xi}(n)\boldsymbol{\xi}^H(n)\right]; \quad \mathbf{p}_\xi = E\left[\boldsymbol{\xi}(n)\right], \tag{4}$$

it is possible to express the unconstrained cost function for the polynomial formulation as follows:

$$\begin{aligned}J_{LB}(\boldsymbol{\theta}) =& E\left[|\upsilon(n) - 1|^2\right] \\ =& 1 - \boldsymbol{\theta}^H\mathbf{p}_\xi - \mathbf{p}_\xi^H\boldsymbol{\theta} + \boldsymbol{\theta}^H\mathbf{R}_\xi\boldsymbol{\theta}.\end{aligned} \tag{5}$$

This function has some important intrinsic characteristics, as presented in the next section.

3. The Blind Equalizability Index

A closer look at Eq. (5) reveals that the unconstrained case directly corresponds to a nonlinear - since $\boldsymbol{\xi}(n)$ is in the Volterra domain - Wiener filtering problem, but the dependence with respect to the free parameters $\boldsymbol{\theta}$ remains linear. This allows us to obtain a closed-form solution that minimizes $J_{LB}(\boldsymbol{\theta})$ as a consequence of the Wiener-Hopf equations [2]:

$$\boldsymbol{\theta}_o = \mathbf{R}_\xi^{-1}\mathbf{p}_\xi. \tag{6}$$

The minimum MSE value can be straightforwardly obtained from Wiener filtering theory [2]:

$$J_{LB}(\boldsymbol{\theta}_o) = 1 - \mathbf{p}_\xi^H\mathbf{R}_\xi^{-1}\mathbf{p}_\xi. \tag{7}$$

Since we abandoned the original constraints, it is expected that $J_{LB}(\boldsymbol{\theta}_o)$ lead to a MSE value lower than or equal to that attained in the constrained case (3), which is an exact representation of the original cost $J_{CM}(\mathbf{w})$. Based on this, we can state that

$$J_{CM}(\mathbf{w}) \geq J_{LB}(\boldsymbol{\theta}_o). \tag{8}$$

Thereby, $J_{LB}(\boldsymbol{\theta}_o)$ can be understood as a lower bound to the CM cost.

As previous works indicated [3, 4, 5], insofar as the connections between CM and Wiener solutions are concerned, $J_{LB}(\boldsymbol{\theta}_o)$ evokes the notion of a blind equalizability index, since it is related to the attainable MSE level, given a certain channel.

In face of recent requirements for greater data rates, it is important to analyze the behavior of the CM lower bound and its associated equalizability index over a broad range of scenarios, including those characterized by multilevel modulated signals. In the next section, we present results for 8-PAM and 16-QAM signals.

4. Simulation Results

In the simulations, we proceeded similarly to [5] for the analysis of the blind equalizability index. To do so, we compared the performance of $J_{LB}(\boldsymbol{\theta}_o)$

to those of the optimal CM and (supervised) Wiener solutions (taking into account the equalization delay). For simulation effects, we estimate the minimum value of the CM cost function by initializing the CMA ($\mu = 0.0008$) at the best Wiener solution (in terms of the equalization delay).

Firstly, in a scenario with real-valued parameters, the channel transfer function is given by $H(z) = 1 + \alpha z^{-1}$ (normalized to have unit norm), where α varies from 0 to 3, and the equalizer is a two-tap filter. The source is composed of 50000 independent and identically distributed ($i.i.d.$) 8-PAM samples and there is no additive noise. The obtained values for $J_{LB}(\boldsymbol{\theta}_o)$ and the minimum value of the CM and Wiener costs are illustrated in Fig. 1.

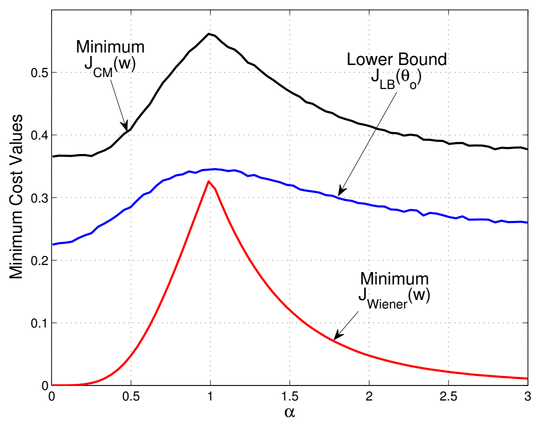

Figure 1. Minimum Cost Values - 8-PAM and first-order channel.

The figure shows that $J_{LB}(\boldsymbol{\theta}_o)$ still holds its functionality as a lower bound for the CM cost. However, due to the many points in signal's constellation, there exists a residual cost offset which raises the curve of the lower bound (and also the minimum CM cost). As a consequence, differently from [3, 4, 5], there is a greater distance between the minimum Wiener cost and $J_{LB}(\boldsymbol{\theta}_o)$. Nonetheless, it is noticeable that there is a smoothed tendency of $J_{LB}(\boldsymbol{\theta}_o)$ to follow the general shape of the other costs, revealing that the idea of equalizability index is preserved.

Assuming now a second-order channel with transfer function $H(z) = 1 + \alpha z^{-1} + \beta z^{-2}$ (with subsequent unit-power normalization), where α and β vary from 0 to 2, we employ a three-tap equalizer. The results, shown in Fig. 2, are a natural extension of the previous case, holding the same observations.

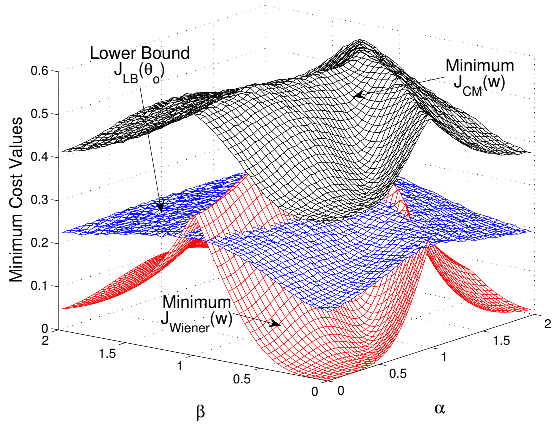

Figure 2. Minimum Cost Values - 8-PAM and second-order channel.

For complex-valued systems and parameters, we assume a complex channel of the type $H(z) = 1 + j\alpha z^{-1}$, with α varying from 0 to 3, an $i.i.d.$ 16-QAM modulated source of 50000 samples without noise and a two-tap complex equalizer. Fig. 3 shows the obtained cost values and the CM lower bound for this scenario.

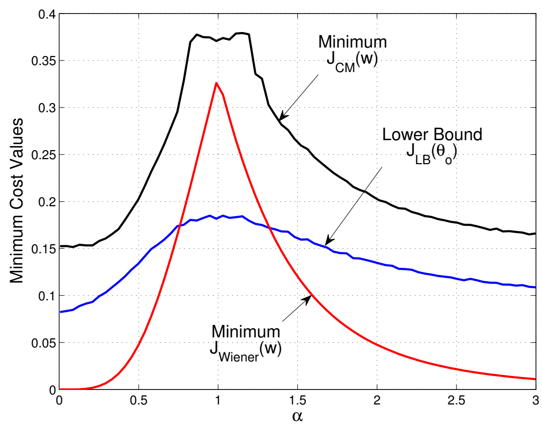

Figure 3. Minimum Cost Values - 16-QAM and first-order complex channel.

In this case, we observe the same phenomenon of the residual error offset in $J_{LB}(\boldsymbol{\theta}_o)$ and in the minimum CM cost – their curves are shifted vertically – which, as the presented figures suggests, is a common characteristic when the modulation is built from a multilevel constellation. Also, it is possible to note that the minimum CM cost, instead of a singular peak, forms a kind of plateau – the reasons underlying this change will be analyzed in the future. Interestingly, the lower bound does not follow this behavior, a consequence of its formulation as a non-linear Wiener filter. In addition to that, it is important to note that $J_{LB}(\boldsymbol{\theta}_o)$, despite the offset, still carries a certain amount of information about the channel equalizability.

As a last test, we keep the same scenario seen in the previous case, but allow an extra degree of freedom for the real part in the complex first-order channel, i.e., we assume the transfer function $H(z) = 1 + (\alpha + j\beta)\, z^{-1}$, with α and β varying from 0 to 2. The result is shown in Fig. 4. It is possible to see that the plateau in the minimum CM cost exists only when the imaginary part is predominant in channel $H(z)$ and the lower bound $J_{LB}(\boldsymbol{\theta}_o)$ is, as the minimum Wiener cost, independent of its presence.

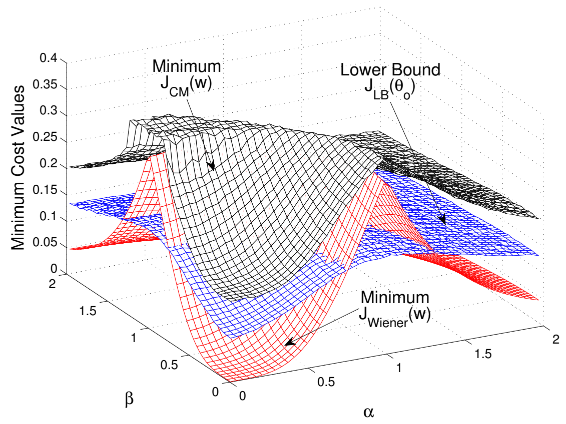

Figure 4. Minimum Cost Values - 16-QAM and first-order complex channel with 2 varying parameters.

A common aspect of all cases is the presence of an offset in the lower bound and in the minimum CM cost as well and, also, that $J_{LB}(\boldsymbol{\theta}_o)$ follows the general behavior of the minimum Wiener cost. Hence, the idea of a blind equalizability index can still be employed for multilevel modulated signals. Moreover, this index can be improved if the offset is set to zero, which can be made, for example, through the use of the information in the figures presented in this work. This, however, remains as a future perspective.

5. Conclusions

In this work, we extended the analysis of the CM lower bound and its interpretation as a blind equalizability index to multilevel modulated signals. In spite of covering only the two specific cases of 8-PAM and 16-QAM signals, some interesting conclusions could be obtained from the results. First, there exists now a residual error offset in the minimum CM cost and in the lower bound, which caused some distancing from the minimum Wiener Cost. Nevertheless, there is a certain consonance in the behavior of the three analyzed minimum costs, contributing positively to the validation of the index in such cases.

As future works, we intend to perform more extensive simulations to signals with larger constellations, like 64- and 128-QAM. Also, we wish to study the plateau for the minimum CM cost in the complex case.

Acknowledgements

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