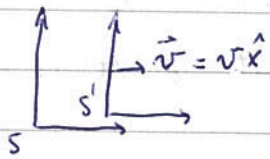
Eletrodinâmica Relativística



$$\vec{\nabla}'\times\vec{E}' = -\frac{\partial \vec{B}'}{\partial t'} \quad , \quad \vec{\nabla}'\times\vec{H}' = \frac{\partial \vec{D}'}{\partial t'} + \vec{J}'$$

$$\vec{\nabla}'\cdot\vec{D}' = \rho' \quad , \quad \vec{\nabla}'\cdot\vec{B}' = 0$$

(notação de índices repetidos) $\quad (\vec{\nabla}'\times\vec{E}')_i = \varepsilon_{ijk}\dfrac{\partial}{\partial x_j'}E_k' = \varepsilon_{ijk}\partial_j' E_k'$

$$\vec{\nabla}'\cdot\vec{B}' = \frac{\partial}{\partial x_i'}B_i' = \partial_i' B_i'$$

Soma implícita, $\quad \delta_{ij}x_j = x_i \quad , \quad \varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$$(\vec{\nabla}'\times\vec{E}')_i = \varepsilon_{ijk}\partial_j' E_k' = -\partial_{t'} B_i' \quad , \quad \vec{\nabla}'\cdot\vec{B}' = \partial_i' B_i' = 0$$

seja $F(x',y',z',t')$: $\quad dF = \dfrac{\partial F}{\partial x'}dx' + \dfrac{\partial F}{\partial y'}dy' + \dfrac{\partial F}{\partial z'}dz' + \dfrac{\partial F}{\partial t'}dt' = \partial_i' F\, dx_i'$

para um dado evento (x',y',z',t') :

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy + \frac{\partial x'}{\partial z}dz + \frac{\partial x'}{\partial t}dt \quad , \text{ etc.}$$

de acordo com as transf. de Lorentz:

$$x' = \gamma(x - vt) \quad , \quad y' = y \quad , \quad z' = z$$

$$t' = \gamma(t - vx/c^2) \quad , \quad \gamma = (1 - v^2/c^2)^{-1/2} = \text{const.}$$

$$\Rightarrow \frac{\partial x'}{\partial x} = \gamma \quad , \quad \frac{\partial x'}{\partial y} = \frac{\partial x'}{\partial z} = 0 \quad , \quad \frac{\partial x'}{\partial t} = -\gamma v \Rightarrow dx' = \gamma\, dx - \gamma v\, dt$$

analogamente: $\quad dy' = dy \quad , \quad dz' = dz$

$$dt' = \frac{\partial t'}{\partial x}dx + \frac{\partial t'}{\partial y}dy + \frac{\partial t'}{\partial z}dz + \frac{\partial t'}{\partial t}dt = -\frac{\gamma v}{c^2}dx + \gamma\, dt$$

Substituindo em dF:

$$dF = \frac{\partial F}{\partial x'}(\gamma\, dx - \gamma v\, dt) + \frac{\partial F}{\partial y'}dy + \frac{\partial F}{\partial z'}dz + \frac{\partial F}{\partial t'}\left(\gamma\, dt - \frac{\gamma v}{c^2}dx\right) =$$

$$= \left(\gamma\frac{\partial F}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial F}{\partial t'}\right)dx + \frac{\partial F}{\partial y'}dy + \frac{\partial F}{\partial z'}dz + \left(\gamma\frac{\partial F}{\partial t'} - \gamma v\frac{\partial F}{\partial x'}\right)dt \Rightarrow$$

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right), \quad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)$$

$$\partial x = \gamma(\partial x' - v/c^2\, \partial t'), \quad \partial y' = \partial y, \quad \partial z' = \partial z, \quad \partial t = \gamma(\partial t' - v\,\partial x')$$

agora, por exemplo: (coord. y)

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \Rightarrow$$

$$\Rightarrow \frac{\partial E_x}{\partial z'} - \gamma\left(\frac{\partial E_z}{\partial x'} - \frac{v}{c^2}\frac{\partial E_z}{\partial t'}\right) = -\gamma\left(\frac{\partial B_y}{\partial t'} - v\frac{\partial B_y}{\partial x'}\right) \Rightarrow$$

$$\Rightarrow \frac{\partial E_x}{\partial z'} - \frac{\partial}{\partial x'}\gamma(E_z + vB_y) = -\frac{\partial}{\partial t'}\gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

em S': $\frac{\partial E_x'}{\partial z'} - \frac{\partial E_z'}{\partial x'} = -\frac{\partial B_y'}{\partial t'} \Rightarrow$

transformações: $\begin{cases} E_x' = E_x \\ E_z' = \gamma(E_z + vB_y) \\ B_y' = \gamma(B_y + v/c^2 E_z) \end{cases}$

agora, da coord z: $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \Rightarrow$

$$\Rightarrow \gamma\left(\frac{\partial E_y}{\partial x'} - \frac{v}{c^2}\frac{\partial E_y}{\partial t'}\right) - \frac{\partial E_x}{\partial y'} = -\gamma\left(\frac{\partial B_z}{\partial t'} - v\frac{\partial B_z}{\partial x'}\right) \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial x'}\gamma(E_y - vB_z) - \frac{\partial E_x}{\partial y'} = -\frac{\partial}{\partial t'}\gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

em S': $\frac{\partial}{\partial x'}E_y' - \frac{\partial E_x'}{\partial y'} = -\frac{\partial B_z'}{\partial t'} \Rightarrow$

transformações: $\begin{cases} E_y' = \gamma(E_y - vB_z) \\ E_x' = E_x \\ B_z' = \gamma(B_z - v/c^2 E_y) \end{cases}$

e da coord. x: $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \Rightarrow$

$$\Rightarrow \frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = -\gamma\left(\frac{\partial B_x}{\partial t'} - v\frac{\partial B_x}{\partial x'}\right) \Rightarrow$$ transf. inversas em E_z e E_y

$$\Rightarrow \gamma\left(\frac{\partial}{\partial y'} E_z' - v\frac{\partial B_y'}{\partial y'}\right) - \gamma\left(\frac{\partial}{\partial z'}E_y' + v\frac{\partial B_z'}{\partial z'}\right) = -\gamma\left(\frac{\partial B_x}{\partial t'} - v\frac{\partial B_x}{\partial x'}\right) \Rightarrow$$

$$\Rightarrow -v\left(\frac{\partial B_x}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'}\right) + \frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} = -\frac{\partial B_x}{\partial t'} \Rightarrow$$

$$\Rightarrow \frac{\partial B_x}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'} = \frac{1}{v}\left(\frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} + \frac{\partial B_x}{\partial t'}\right) \qquad (1)$$

mas de $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \Rightarrow$$

$$\gamma\left(\frac{\partial B_x}{\partial x'} - \frac{v}{c^2}\frac{\partial B_x}{\partial t'}\right) + \frac{B_y}{\partial y'} + \frac{\partial B_z}{\partial z'} = 0 \Rightarrow$$ transf. inversas em B_y e B_z:

$$\gamma\left(\frac{\partial B_x}{\partial x'} - \frac{v}{c^2}\frac{\partial B_x}{\partial t'}\right) + \gamma\left(\frac{\partial B_y'}{\partial y'} - \frac{v}{c^2}\frac{\partial E_z'}{\partial y'}\right) + \gamma\left(\frac{\partial B_z'}{\partial y'} + \frac{v}{c^2}\frac{\partial E_y'}{\partial z'}\right) = 0 \Rightarrow$$

$$\frac{\partial B_x}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'} = \frac{v}{c^2}\left(\frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} + \frac{\partial B_x}{\partial t'}\right) \qquad (2)$$

em S': $\frac{\partial B_x'}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'} = 0$, $\frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} = -\frac{\partial B_x'}{\partial t'}$

(1) e (2) $\Rightarrow$ $B_x' = B_x$

completando as transformações:

$E_x' = E_x$	$E_x = E_x'$
$E_y' = \gamma(E_y - vB_z)$	$E_y = \gamma(E_y' + vB_z')$
$E_z' = \gamma(E_z + vB_y)$	$E_z = \gamma(E_z' - vB_y')$
$B_x' = B_x$	$B_x = B_x'$
$B_y' = \gamma(B_y + v/c^2 E_z)$	$B_y = \gamma(B_y' - v/c^2 E_z')$
$B_z' = \gamma(B_z - v/c^2 E_y)$	$B_z = \gamma(B_z' + v/c^2 E_y')$

def. do tensor (de rank-2, antissimétrico) intensidade de campo :

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

e do tensor dual $G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$

as equações de Maxwell tornam-se:

1) $\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$, 2) $\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$

1) as equações homogêneas:

$$\partial_\nu G^{\mu\nu} = \frac{\partial G^{\mu\nu}}{\partial x^\nu} = \underbrace{0^\mu}_{\text{(4 componentes)}} = (0, \vec{0}) = 0$$

para $\mu = 0$: $\frac{\partial G^{0\nu}}{\partial x^0} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} =$

$$= \frac{\partial 0}{c\partial t} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

para $\mu = 1$: $\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} =$

$$= -\frac{\partial B_x}{c\partial t} + 0 - \frac{\partial E_z}{c\partial y} + \frac{\partial E_y}{c\partial z} = -\frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times\vec{E}\right)_x = 0$$

analogamente: $\mu = 2$: $\frac{\partial G^{2\nu}}{\partial x^\nu} = -\frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times\vec{E}\right)_y = 0$

$\mu = 3$: $\frac{\partial G^{3\nu}}{\partial x^\nu} = -\frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times\vec{E}\right)_z = 0$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

2) equações não-homogêneas

def.: quadrivetor densidade de corrente $J^\mu = (c\rho, \vec{J})$

lembrando que $\vec{J} = \rho\vec{v}$ e que $V = \gamma(c, \vec{v})$

agora $\frac{\partial J^\mu}{\partial x^\mu} = 0$ é a equação da continuidade:

$$\partial_\mu J^\mu = \frac{\partial J^\mu}{\partial x^\mu} = \frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = \frac{\cancel{c}\,\partial\rho}{\cancel{c}\,\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$$

$$\partial_\nu F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$$

para $\mu = 0$: $\displaystyle \frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} =$

$$= 0 + \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}\vec{\nabla}\cdot\vec{E}$$

$$\mu_0 J^0 = \mu_0 c\rho$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \mu_0 c^2 \rho = \frac{\cancel{\mu_0}}{\varepsilon_0 \cancel{\mu_0}}\rho = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0}$$

para $\mu = 1$: $\displaystyle \frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} =$

$$= -\frac{\partial E_x}{c^2\partial t} + 0 + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x$$

analogamente para $\mu = 2, 3$: $\displaystyle \frac{\partial F^{2\nu}}{\partial x^\nu} = \left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_y$, $\displaystyle \frac{\partial F^{3\nu}}{\partial x^\nu} = \left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_z$

para $\mu = 1, 2, 3$: $\mu_0 J^\mu = \mu_0 J_x , \mu_0 J_y , \mu_0 J_z$

$$\Rightarrow -\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}$$

$$\Rightarrow \boxed{\partial_\nu G^{\mu\nu} = 0} , \boxed{\partial_\nu F^{\mu\nu} = \mu_0 J^\mu}$$

As equações de Maxwell ficam:

1) $\vec{\nabla}'\cdot\vec{E}' = \frac{\rho'}{\varepsilon_0}$, $\quad c\rho' = \gamma(c\rho - vJ_x/c) \Rightarrow \rho' = \gamma(\rho - vJ_x\mu_0\varepsilon_0)$

$$E_x' = E_x$$
$$E_y' = \gamma(E_y - vB_z) \qquad \Big| \quad \partial_x' = \gamma\left(\partial_x + \frac{v}{c^2}\partial_t\right)$$
$$E_z' = \gamma(E_z + vB_y) \qquad \Big| \quad \partial_y' = \partial_y \,,\ \partial_z' = \partial_z$$

$$\vec{\nabla}'\cdot\vec{E}' = \partial_x'E_x' + \partial_y'E_y' + \partial_z'E_z' = \gamma\left(\partial_x E_x + \frac{v}{c^2}\partial_t E_x\right) +$$
$$+ \gamma(\partial_y E_y - v\,\partial_y B_z) + \gamma(\partial_z E_z + v\,\partial_z B_y) =$$
$$= \gamma(\vec{\nabla}\cdot\vec{E}) + \gamma v\Big(\frac{1}{c^2}\partial_t E_x \underbrace{- \partial_y B_z + \partial_z B_y}_{(\vec{\nabla}\times\vec{B})_x}\Big) =$$
$$= \gamma\frac{\rho}{\varepsilon_0} + \gamma v(-\mu_0 J_x) =$$
$$= \gamma\frac{\rho}{\varepsilon_0} - \gamma v J_x \mu_0 \frac{\varepsilon_0}{\varepsilon_0} = \frac{\gamma(\rho - vJ_x\mu_0\varepsilon_0)}{\varepsilon_0} = \frac{\rho'}{\varepsilon_0}$$

2) $$\vec{\nabla}'\cdot\vec{B}' = \partial_x'B_x' + \partial_y'B_y' + \partial_z'B_z' = \gamma\left(\partial_x B_x + \frac{v}{c^2}\partial_t B_x\right) +$$
$$+ \gamma\left(\partial_y B_y + \frac{v}{c^2}\partial_y E_z\right) + \gamma\left(\partial_z B_z - \frac{v}{c^2}\partial_z E_y\right) =$$
$$= \gamma\,\underbrace{\vec{\nabla}\cdot\vec{B}}_{\to 0} + \frac{\gamma v}{c^2}\Big(\partial_t B_x + \underbrace{\partial_y E_z - \partial_z E_y}_{(\vec{\nabla}\times\vec{E})_x = -\partial_t B_x}\Big)_{\to 0} = 0$$

3) $$(\vec{\nabla}'\times\vec{E}')_x = \partial_y'E_z' - \partial_z'E_y' =$$
$$= \gamma(\partial_y E_z + v\,\partial_y B_y) - \gamma(\partial_z E_y - v\,\partial_z B_z) =$$
$$= \gamma(\partial_y E_z - \partial_z E_y) + \gamma v(\partial_y B_y + \partial_z B_z) =$$
$$= \gamma(\vec{\nabla}\times\vec{E})_x + \gamma v(-\partial_x B_x) = -\gamma\,\partial_t B_x - \gamma v\,\partial_x B_x$$

mas $\partial_t' B_x' = \gamma(\partial_t + v\,\partial_x)B_x = \gamma\,\partial_t B_x + \gamma\,\partial_x B_x$

$$\therefore\ (\vec{\nabla}\times\vec{E})_x = -\partial_t' B_x'$$

4) $\vec{\nabla}' \times \vec{B}' = \mu_0 \vec{J}' + \frac{1}{c^2}\frac{\partial \vec{E}'}{\partial t'}$, $J_x' = \gamma(J_x - \frac{v}{c} c\rho)$

$\partial t' = \gamma(\partial t + v\, \partial x)$

$$(\vec{\nabla}' \times \vec{B}')_x = \partial_y' B_z' - \partial_z' B_y' =$$
$$= \gamma(\partial_y B_z - v/c^2\, \partial_y E_y) - \gamma(\partial_z B_y + v/c^2\, \partial_z E_z) =$$
$$= \gamma(\partial_y B_z - \partial_z B_y) - \gamma v/c^2 (\partial_y E_y + \partial_z E_z) =$$
$$= \gamma(\vec{\nabla} \times \vec{B})_x - \gamma v/c^2 (\rho/\varepsilon_0 - \partial_x E_x) =$$
$$= \gamma \mu_0 J_x + (\gamma/c^2)\partial_t E_x + (\gamma v/c^2)\partial_x E_x - \underbrace{\gamma v \rho / c^2 \varepsilon_0}_{-\gamma v \rho \mu_0} =$$
$$= \mu_0 \gamma(J_x - v\rho) + \gamma/c^2 (\partial_t E_x + v\, \partial_x E_x) =$$
$$= \mu_0 J_x' + \frac{1}{c^2} \partial_t' E_x'$$