

Classical Electrodynamics III

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Ref.: [1] J. D. Jackson, *Classical Electrodynamics*,
3rd Ed., John Wiley & Sons (1999).

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Chapter 11 – Special Theory of Relativity

1. Situation before 1900, Einstein's Two Postulates

Maxwell's equations

(1864) Maxwell's equations: electromagnetism and optics

- Wave theory: a medium for the propagation of waves (ether).
- Ether: rarefied medium supposed to have no interaction with matter and to permeate all space, through which electromagnetic waves propagate. [Aristotle (4th century BC), Descartes (17th century), Fresnel (19th century)].

Electromagnetism versus Mechanics

Consider two reference frames, K and K' , with relative velocity $\vec{v}$.

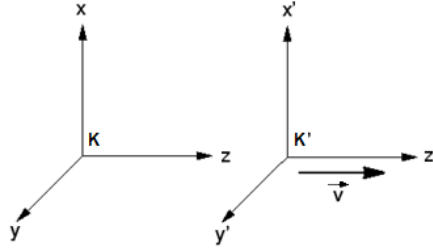


Figure 1: Illustration of two reference frames, K and K' , with relative velocity $\vec{v}$.

The Galilean transformations are

$$\vec{x}' = \vec{x} - \vec{v}t, \quad t' = t. \quad (1)$$

For a system of particles interacting through a central potential, in the reference frame K' ,

$$m_i \frac{d^2 \vec{v}'_i}{dt'^2} = -\vec{\nabla}'_i \sum_j V_{ij} (|\vec{x}'_i - \vec{x}'_j|). \quad (2)$$

From Eq. (1),

$$\vec{v}'_i = \vec{v}_i - \vec{v}, \quad \vec{\nabla}'_i = \vec{\nabla}_i, \quad \frac{d\vec{v}_i}{dt'} = \frac{d\vec{v}_i}{dt}, \quad \text{and} \quad |\vec{x}'_i - \vec{x}'_j| = |\vec{x}_i - \vec{x}_j|.$$

Therefore, in the reference frame K ,

$$m_i \frac{d^2 \vec{x}_i}{dt^2} = -\vec{\nabla}_i \sum_j V_{ij}(|\vec{x}_i - \vec{x}_j|). \quad (3)$$

Consider a field $\psi(\vec{x}', t')$ satisfying the equation

$$\left(\vec{\nabla}'^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \right) \psi = 0. \quad (4)$$

First, we know that

$$d\psi = \sum_i \frac{\partial \psi}{\partial x'_i} dx'_i + \frac{\partial \psi}{\partial t'} dt'.$$

But, from Eq. (1),

$$d\vec{x}' = d\vec{x} - \vec{v} dt, \quad dx'_i = dx_i - v_i dt, \quad dt' = dt.$$

Thus,

$$d\psi = \sum_i \frac{\partial \psi}{\partial x'_i} dx_i + \left[\frac{\partial \psi}{\partial t'} - \vec{v} \cdot \vec{\nabla}' \psi \right] dt.$$

Therefore,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \vec{v} \cdot \vec{\nabla}'.$$

Hence,

$$\frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial t'} - \vec{v} \cdot \vec{\nabla}' \right)^2.$$

Similarly,

$$\frac{\partial^2}{\partial t'^2} = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right)^2.$$

Consequently,

$$\left[\vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{2}{c^2} \vec{v} \cdot \vec{\nabla} \frac{\partial}{\partial t} - \frac{1}{c^2} (\vec{v} \cdot \vec{\nabla})^2 \right] \psi = 0. \quad (5)$$

The wave equation is not preserved in the form of the Galilean transformations.

Possibilities

1. Maxwell's equations were incorrect. The proper theory of electromagnetism should be invariant under Galilean transformations.
2. Galilean relativity applied to mechanics and there exists a privileged inertial frame in which the ether is at rest.
3. Under another kind of transformations between inertial frames, the electromagnetic theory has the same form in every inertial reference frame.

Discussion

1. Difficult to accept: success of Maxwell's equations at the hands of Hertz, Lorentz, *et al.*
2. Acceptable, but Michelson–Morley (1887) obtained a null result for the motion of the Earth relative to the ether.

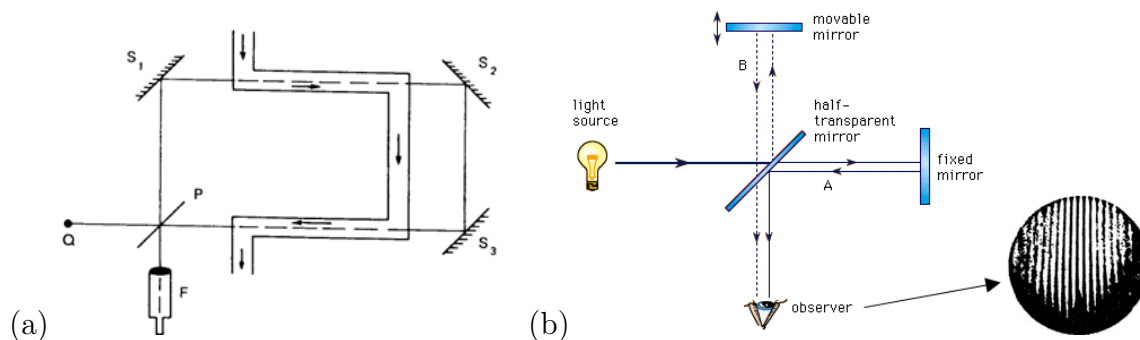


Figure 2: (a) The Fizeau experiment; (b) the Michelson-Morley interferometer.

(1892) Introduction of the Fitzgerald–Lorentz contraction:

$$L(v) = L_0 \sqrt{1 - \frac{v^2}{c^2}}. \quad (6)$$

Other experiments also failed: Fizeau experiments (1851, 1853) and later Michelson and Morley (1886) experiments on the velocity of light in moving fluids could be only understood if one supposed the ether was dragged along partially by the moving fluid.

3. Lorentz and Poincaré showed that Maxwell's equations are invariant under Lorentz transformations — and Eq. (6) is one of their consequences. Einstein chose this possibility.

Postulates of the Theory of Relativity

1. Postulate of relativity:

The laws of physics are the same in all inertial reference frames.

2. Postulate of the constancy of the velocity of light:

The velocity of light is finite and independent of the motion of the source relative to the observer.

2'. Postulate of a universal limiting velocity:

In every inertial reference frame there is a finite and universal limiting velocity c for all physical entities.

2. Recent Experiments

A) Ether Drift

The null result of the test of Michelson and Morley (1887) established that

$$v_{drift} < \frac{1}{3}v_{orb} \sim 10^4 \text{ m/s.}$$

A review is given by [2].

(1958) Mössbauer effect:

Suppose we have a gamma-ray emitter nucleus:

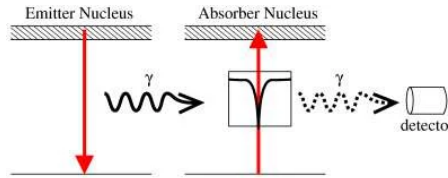


Figure 3: Emission and resonant absorption of gamma rays.

The energy of the excitation should be splitted between the photon and the nucleus:

$$E_0 = E_\gamma + \frac{p_\gamma^2}{2M} = E_\gamma + \frac{E_\gamma^2}{2Mc^2}.$$

Solving for E_γ , disregarding the negative solution and expanding to the 3rd order:

$$E_\gamma = Mc^2 \left[-1 + \sqrt{1 + \frac{2E_0}{Mc^2}} \right] \simeq E_0 - \frac{E_0^2}{2Mc^2}.$$

But, in the Mössbauer effect, the recoil is taken up by the solid:

$$E_\gamma \simeq E_0,$$

giving astounding precision in frequency with no Doppler shifts.

Doppler Effect

The phase of a plane wave is invariant (its value is constant in different inertial reference frames). For a plane electromagnetic wave, $c = \lambda f$, and

$$\phi = \omega t - \vec{k} \cdot \vec{x} = \omega \left(t - \frac{k}{\omega} \hat{n} \cdot \vec{x} \right) = \omega \left(t - \frac{\hat{n} \cdot \vec{x}}{c} \right).$$

Thus, in another reference frame,

$$\phi = \omega' \left(t' - \frac{\hat{n}' \cdot \vec{x}'}{c'} \right). \quad (7)$$

From Eq. (1), one obtains

$$\omega \left[t' \left(1 - \frac{\hat{n} \cdot \vec{v}}{c} \right) - \frac{\hat{n} \cdot \vec{x}'}{c} \right] = \omega' \left[t' - \frac{\hat{n}' \cdot \vec{x}'}{c'} \right].$$

Since this must be valid for every t' and $\vec{x}'$,

$$\hat{n} = \hat{n}', \quad \omega' = \omega \left(1 - \frac{\hat{n} \cdot \vec{v}}{c} \right), \quad c' = c - \hat{n} \cdot \vec{v}. \quad (8)$$

These are the standard Doppler shift formulas of Galilean relativity.

A geometrical construction is then given. Let us watch a plane wave propagate through 2 inertial reference frames K and K'. At $t_0 = 0$ a given crest is at the point A and after an unitary time, $t = 1$, it arrives at the point B in K (or B' in K').

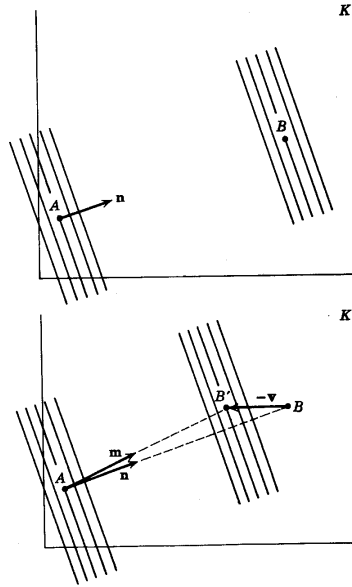


Figure 4: A plane wave in two reference frames.

If the distance between A and B is c , then

$$\vec{m} = c\hat{n} - \vec{v},$$

and its versor is

$$\hat{m} = \frac{c\hat{n} - \vec{v}}{|c\hat{n} - \vec{v}|}. \quad (9)$$

For small v/c ,

$$|c\hat{n} - \vec{v}| = c|\hat{n} - \vec{v}/c| \simeq c \left(1 - \frac{\hat{n} \cdot \vec{v}}{c} \right).$$

This result is similar to

$$\sim c \left(1 - \frac{\hat{m} \cdot \vec{v}_0}{c} \right),$$

where v_0 is the laboratory speed relative to the ether. Consequently,

$$\hat{m} \left(1 - \frac{\hat{m} \cdot \vec{v}_0}{c} \right) \simeq \hat{n} - \frac{\vec{v}_0}{c}.$$

Hence,

$$\hat{n} \simeq \left(1 - \frac{\hat{m} \cdot \vec{v}_0}{c} \right) \hat{m} + \frac{\vec{v}_0}{c}. \quad (10)$$

Now, let a plane wave be observed by two observers, with frequency ω_0 in the laboratory frame and ω_1 in a frame K_1 with velocity $\vec{v}_1 = \vec{u}_1 + \vec{v}_0$ relatively to the ether. From Eq. (8),

$$\omega_1 = \omega \left(1 - \frac{\hat{n} \cdot \vec{v}_1}{c} \right),$$

and

$$\omega_0 = \omega \left(1 - \frac{\hat{n} \cdot \vec{v}_0}{c} \right).$$

Therefore,

$$\omega_1 = \frac{\omega_0}{1 - \hat{n} \cdot \vec{v}_0/c} \left(1 - \frac{\hat{n} \cdot \vec{v}_1}{c} \right).$$

To first order,

$$\begin{aligned} \omega_1 &\simeq \omega_0 \left(1 - \frac{\hat{n} \cdot \vec{v}_1}{c} \right) \left(1 + \frac{\hat{n} \cdot \vec{v}_0}{c} \right) \Rightarrow \\ \omega_1 &\simeq \omega_0 \left[1 - \frac{\hat{n} \cdot (\vec{v}_1 - \vec{v}_0)}{c} \right] \Rightarrow \\ \omega_1 &\simeq \omega_0 \left[1 - \frac{\hat{u}_1}{c} \cdot \left(\hat{m} + \frac{\vec{v}_0}{c} \right) \right]. \end{aligned} \quad (11)$$

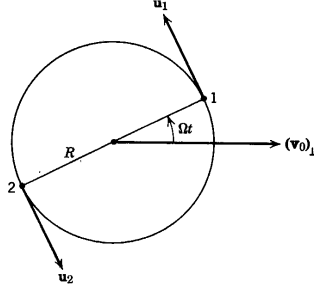


Figure 5: Two Mössbauer systems.

Two Mössbauer systems

Consider two systems, an emitter and an absorber, in the opposite ends of a rod and rotating about a center with angular velocity Ω .

From Eq. (11),

$$\frac{\omega_1 - \omega_2}{\omega_0} = \left(\frac{\vec{u}_2 - \vec{u}_1}{c} \right) \cdot \left(\hat{m} + \frac{\vec{v}_0}{c} \right).$$

Here

$$(\vec{u}_2 - \vec{u}_1) \cdot \hat{m} = 0,$$

such that one obtains

$$\frac{\omega_1 - \omega_2}{\omega_0} = \frac{(\vec{u}_2 - \vec{u}_1) \cdot \vec{v}_0}{c^2} = \frac{2\Omega R \hat{u} \cdot \vec{v}_0}{c^2},$$

and since $\vec{u}_1$ and $\vec{u}_2$ are in opposite directions in the rotating apparatus,

$$|\vec{u}_2 - \vec{u}_1| \simeq 2u = 2 \left(\frac{2\pi R}{T} \right) = 2\Omega R.$$

Taking the projection of $\vec{v}_0$ in the plane perpendicular to the rotation axis (same of $\vec{u}_1, \vec{u}_2$),

$$\frac{\omega_1 - \omega_2}{\omega_0} = \frac{2\Omega R}{c^2} \sin \Omega t |(\vec{v}_0)_\perp|. \quad (12)$$

(1963) Birmingham:

The Mössbauer line of 14.4 keV gamma ray from ^{57}Fe , following the beta decay of ^{57}Co , has been measured in an apparatus as in Fig.5, with

$$R \simeq 4 \text{ cm}, \quad \Omega \simeq 6000 \text{ s}^{-1}, \quad v_0 \simeq 200 \text{ m/s},$$

which gives a sensitivity of approximately

$$\frac{\Delta\omega}{\omega_0} \sim 10^{-12}.$$

No effect was observed to $|(\vec{v}_0)_\perp| = 1.6 \pm 2.8 \text{ m/s}$. A later experiment (1970) gave a limit velocity smaller than approximately 5 cm/s.

(1958) Ammonia masers:

Same concept, should have a beat frequency (under rotation of 180°):

$$\frac{\Delta\omega}{\omega_0} = 4|\vec{u}_{mol} \cdot \vec{v}_0|/c^2,$$

but no ether drift has been measured at less than 30 m/s.

B) Velocity of light from a moving source

Many attempts have been made to make Maxwell's equations not imply that the velocity of light is constant relative to the source.

The **Ritz's electrodynamics**: in his theory the two homogeneous Maxwell equations are kept intact, but the equations evolving sources are modified in such a way that the speed of light was equal to c only when measured relative to the source. The Ritz theory is even in accordance with the following experimental evidences:

- stellar position aberration;
- Fizeau experiment;
- original¹ Michelson–Morley experiment.

Unfortunately, it seems clear that most of the early evidence is invalid because of the interaction of the radiation with the matter through which it passes before detection. Refs.: [3, 4, 5, 6]

(1964) CERN:

A measurement involving the velocity of 6 GeV photons produced in the decay

$$\pi^0 \longrightarrow \gamma\gamma$$

by time of flight over 80 m. The pions were produced by the bombardment of a beryllium target by 19.2 GeV protons and had $\beta \simeq 0.99975$. The velocity of light was found to be c , independent of the motion of the source. If we had

$$c' = c + \kappa v,$$

where v is the speed of the source, then, within the experimental precision

$$\kappa = (0.0 \pm 1.3) \times 10^{-4}.$$

Ref. [7].

Other experiments on charged particles and neutrinos independently establish the validity of the second postulate [8].

C) Frequency dependence of the speed of light in vacuum

The speed of light is known to an accuracy of $\sim 1/10^9$ and the meter is defined to this precision². Suppose that the velocity of light depends on frequency:

$$c(\omega) = c \left(1 - \frac{\omega_0^2}{\omega^2} \right)^{1/2}. \quad (13)$$

Possibly, this can be related to a non-zero photon mass ($E = \hbar\omega_0 = mc^2$).

¹It is ruled out with extraterrestrial light from the sun or other stars.

²By the time when Jackson wrote his book, this was the case, see below ...

The existence of normal modes in the Earth-ionosphere resonant cavity sets a limit of

$$\omega_0 < 10 \, c/R,$$

where R is the radius of the Earth. From radio frequencies to $\omega \rightarrow \infty$, the change due to a photon mass is less than $\Delta c/c \simeq 10^{-10}$.

The dispersion of vacuum (outside special relativity) was tested by pulsars observations over 13 decades of frequency. For two frequencies,

$$\omega_1 < \omega_2,$$

the dispersion in the velocity can be estimated from

$$\left| \frac{c(\omega_1) - c(\omega_2)}{c} \right| \leq \frac{c\Delta t}{D},$$

where Δt is the pulse duration and D is the distance to the pulsar. The quite small time duration of the pulses permits a simple estimate. For the Crab pulsar Np 0532,

$$\Delta t \simeq 3 \times 10^{-3} \text{ s}, \quad D \simeq 6 \times 10^3 \text{ ly}.$$

Thus,

$$\frac{c\Delta t}{D} \simeq 1.7 \times 10^{-14}.$$

Various observations from $\nu \simeq 4 \times 10^{14}$ Hz, through the optical region, and up to $E_\gamma \sim 1$ MeV indicate

$$\frac{\Delta c}{c} < 10^{-14}.$$

(1973) Stanford Linear Accelerator:

For higher energies, experiments involving photons of approximately 7 GeV give the result

$$\frac{\Delta c}{c} < 10^{-5}.$$

Refs.: [9, 10]

(1983) General Conference on Weights and Measures:

The value of the speed of light was fixed as

$$\boxed{c = 299\,792\,458 \text{ m/s}} \quad (\text{exact}).$$

And by definition:

“1 meter is the distance traveled by the light during a time of $\Delta t = 1/c$ seconds”.

3. Lorentz Transformations and Relativistic Kinematics

A) Lorentz Coordinate Transformations

Consider two reference frames K and K' moving with relative velocity $\vec{v}$.

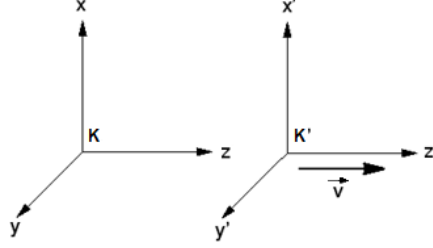


Figure 6: Illustration of two reference frames, K and K' , with relative velocity $\vec{v}$.

The origins of the coordinate systems coincide:

$$O = O' \quad \text{at} \quad t = t' = 0.$$

A light pulse emitted from the common origin must propagate with velocity c in both reference frames. According to Einstein's second postulate, the propagation of light satisfies, in K ,

$$c^2 t^2 - (x^2 + y^2 + z^2) = 0, \quad (14)$$

and, in K' ,

$$c^2 t'^2 - (x'^2 + y'^2 + z'^2) = 0. \quad (14')$$

Since spacetime is homogeneous and isotropic, there must be a linear relation between the two reference frames:

$$c^2 t'^2 - (x'^2 + y'^2 + z'^2) = \lambda^2 [c^2 t^2 - (x^2 + y^2 + z^2)], \quad (15)$$

where $\lambda = \lambda(v)$. The inverse transformation corresponds to $\vec{v} \rightarrow -\vec{v}$ and implies that $\lambda(v) = 1$.

Now consider the case in which $\vec{v} = v\hat{z}$. For a light ray,

$$\begin{cases} z = ct = \gamma(z' + vt') = \gamma(c + v)t', \\ z' = ct' = \gamma(z - vt) = \gamma(c - v)t. \end{cases}$$

Eliminating t and t' ,

$$t' = \frac{ct}{\gamma(c + v)} = \frac{\gamma(c - v)t}{c}.$$

Therefore,

$$\gamma^2 = \frac{c^2}{(c + v)(c - v)} = \frac{c^2}{(c^2 - v^2)}$$

and hence

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c}. \quad (16)$$

Solving for the time coordinate,

$$ct' = \gamma(c - v)t = \gamma(ct - \beta ct).$$

Define

$$x_0 = ct, \quad x_1 = z, \quad x_2 = x, \quad x_3 = y.$$

Then the Lorentz transformations can be written as

$$\begin{cases} x'_0 = \gamma(x_0 - \beta x_1), \\ x'_1 = \gamma(x_1 - \beta x_0), \\ x'_2 = x_2, \\ x'_3 = x_3, \end{cases} \quad (17)$$

where

$$\vec{\beta} = \frac{\vec{v}}{c}, \quad \beta = |\vec{\beta}|, \quad \gamma = (1 - \beta^2)^{-1/2}. \quad (16')$$

The inverse transformation is

$$\begin{cases} x_0 = \gamma(x'_0 + \beta x'_1), \\ x_1 = \gamma(x'_1 + \beta x'_0), \\ x_2 = x'_2, \\ x_3 = x'_3. \end{cases} \quad (18)$$

Now consider a boost in an arbitrary direction. The spatial vector is decomposed into components parallel and perpendicular to $\vec{\beta}$:

$$\vec{x} = \vec{x}_{\parallel} + \vec{x}_{\perp}.$$

The parallel component and the perpendicular component are

$$\vec{x}_{\parallel} = \frac{(\vec{\beta} \cdot \vec{x})}{\beta^2} \vec{\beta}, \quad \vec{x}_{\perp} = \vec{x} - \frac{(\vec{\beta} \cdot \vec{x})}{\beta^2} \vec{\beta}.$$

The transformation of the time component is

$$x'_0 = \gamma(x_0 - \beta x_{\parallel}) = \gamma(x_0 - \vec{\beta} \cdot \vec{x}).$$

For the spatial components,

$$\vec{x}'_{\parallel} = \gamma(\vec{x}_{\parallel} - \vec{\beta} x_0), \quad \vec{x}'_{\perp} = \vec{x}_{\perp}.$$

Since

$$\vec{x}' = \vec{x}'_{\parallel} + \vec{x}'_{\perp},$$

we obtain

$$\vec{x}' = \gamma \left[\frac{(\vec{\beta} \cdot \vec{x})}{\beta^2} \vec{\beta} - \vec{\beta} x_0 \right] + \left[\vec{x} - \frac{(\vec{\beta} \cdot \vec{x})}{\beta^2} \vec{\beta} \right].$$

Hence,

$$\boxed{\vec{x}' = \vec{x} + (\gamma - 1) \frac{(\vec{\beta} \cdot \vec{x})}{\beta^2} \vec{\beta} - \gamma \vec{\beta} x_0} \quad (19)$$

together with

$$\boxed{x'_0 = \gamma (x_0 - \vec{\beta} \cdot \vec{x})}. \quad (19')$$

For $0 \leq \beta < 1 \Leftrightarrow 1 \leq \gamma < \infty$, with $\gamma = (1 - \beta^2)^{-1/2}$.

Boost parameter/Rapidity

Define the rapidity ζ by

$$\beta = \tanh \zeta = \frac{e^\zeta - e^{-\zeta}}{e^\zeta + e^{-\zeta}} = \frac{\sinh \zeta}{\cosh \zeta}.$$

Therefore,

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - \tanh^2 \zeta}} = \cosh \zeta.$$

Thus,

$$\gamma \beta = \sinh \zeta. \quad (20)$$

A boost along the x_1 direction can then be written as

$$\begin{cases} x'_0 = \gamma x_0 - \gamma \beta x_1, \\ x'_1 = \gamma x_1 - \gamma \beta x_0. \end{cases}$$

Using the rapidity,

$$\begin{cases} x'_0 = \cosh \zeta x_0 - \sinh \zeta x_1, \\ x'_1 = -\sinh \zeta x_0 + \cosh \zeta x_1, \\ x'_2 = x_2, \\ x'_3 = x_3. \end{cases} \quad (21)$$

Therefore,

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \cosh \zeta & -\sinh \zeta & 0 & 0 \\ -\sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

which suggests a geometrical interpretation: the Lorentz transformation corresponds to a hyperbolic rotation:

$$x'_\mu = \sum_{\nu=0}^3 (\Lambda_{\nu\mu}) x_\nu,$$

where Λ is the Lorentz transformation matrix.

B) Four-vectors

Vectors of four dimensions that represent space and time coordinates of an event,

$$(A_0, A_1, A_2, A_3) = (A_0, \vec{A}),$$

are called four-vectors. Their components transform according to

$$\begin{cases} A'_0 = \gamma (A_0 - \vec{\beta} \cdot \vec{A}), \\ A'_\parallel = \gamma (A_\parallel - \beta A_0), \\ \vec{A}'_\perp = \vec{A}_\perp, \end{cases} \quad (22)$$

or,

$$\begin{cases} A'_0 = \gamma (A_0 - \vec{\beta} \cdot \vec{A}), \\ \vec{A}' = \vec{A} + (\gamma - 1) \frac{(\vec{\beta} \cdot \vec{A})}{\beta^2} \vec{\beta} - \gamma \vec{\beta} A_0. \end{cases} \quad (22')$$

From the spacetime invariance

$$c^2 t'^2 - |\vec{x}'|^2 = c^2 t^2 - |\vec{x}|^2,$$

follows that

$$A_0'^2 - |\vec{A}'|^2 = A_0^2 - |\vec{A}|^2. \quad (23)$$

The pairs $(A_0, \vec{A})$ and $(A'_0, \vec{A}')$ are, respectively, the components of the same four-vector in the reference frames K and K' .

For two four-vectors A and B , the “scalar product” is invariant:

$$A' \cdot B' = A'_0 B'_0 - \vec{A}' \cdot \vec{B}' = A_0 B_0 - \vec{A} \cdot \vec{B} = A \cdot B. \quad (24)$$

Writing

$$\vec{A}' = \vec{A}'_\parallel + \vec{A}'_\perp, \quad \vec{B}' = \vec{B}'_\parallel + \vec{B}'_\perp,$$

follows that

$$A' \cdot B' = A'_0 B'_0 - (\vec{A}'_\parallel + \vec{A}'_\perp) \cdot (\vec{B}'_\parallel + \vec{B}'_\perp) = A'_0 B'_0 - \vec{A}'_\parallel \cdot \vec{B}'_\parallel - \vec{A}'_\perp \cdot \vec{B}'_\perp.$$

Now, the Lorentz transformation gives

$$\begin{aligned} A'_0 &= \gamma (A_0 - \vec{\beta} \cdot \vec{A}), & B'_0 &= \gamma (B_0 - \vec{\beta} \cdot \vec{B}), \\ \vec{A}'_\parallel &= \gamma (\vec{A}_\parallel - \vec{\beta} A_0), & \vec{A}'_\perp &= \vec{A}_\perp, \\ \vec{B}'_\parallel &= \gamma (\vec{B}_\parallel - \vec{\beta} B_0), & \vec{B}'_\perp &= \vec{B}_\perp. \end{aligned}$$

Therefore, the space part is

$$\begin{aligned} \vec{A}' \cdot \vec{B}' &= \vec{A}'_\parallel \cdot \vec{B}'_\parallel + \vec{A}'_\perp \cdot \vec{B}'_\perp \\ &= \gamma^2 (\vec{A}_\parallel - \vec{\beta} A_0) \cdot (\vec{B}_\parallel - \vec{\beta} B_0) + \vec{A}_\perp \cdot \vec{B}_\perp \\ &= \gamma^2 [\vec{A}_\parallel \cdot \vec{B}_\parallel - A_0 (\vec{\beta} \cdot \vec{B}_\parallel) - B_0 (\vec{\beta} \cdot \vec{A}_\parallel) + \beta^2 A_0 B_0] + \vec{A}_\perp \cdot \vec{B}_\perp. \end{aligned}$$

Since $\vec{\beta} \cdot \vec{A}_\perp = 0$, and $\vec{\beta} \cdot \vec{B}_\perp = 0$, we have

$$\vec{\beta} \cdot \vec{A}_\parallel = \vec{\beta} \cdot \vec{A}, \quad \vec{\beta} \cdot \vec{B}_\parallel = \vec{\beta} \cdot \vec{B}.$$

Hence,

$$\begin{aligned} A'_0 B'_0 - \vec{A}' \cdot \vec{B}' &= \gamma^2 \left(A_0 - \vec{\beta} \cdot \vec{A} \right) \left(B_0 - \vec{\beta} \cdot \vec{B} \right) \\ &\quad - \gamma^2 \left[\vec{A}_\parallel \cdot \vec{B}_\parallel - A_0 (\vec{\beta} \cdot \vec{B}) - B_0 (\vec{\beta} \cdot \vec{A}) + \beta^2 A_0 B_0 \right] \\ &\quad - \vec{A}_\perp \cdot \vec{B}_\perp. \end{aligned}$$

Expanding the first term,

$$\begin{aligned} A'_0 B'_0 - \vec{A}' \cdot \vec{B}' &= \gamma^2 \left[A_0 B_0 - \cancel{A_0 (\vec{\beta} \cdot \vec{B})} - \cancel{B_0 (\vec{\beta} \cdot \vec{A})} + (\vec{\beta} \cdot \vec{A})(\vec{\beta} \cdot \vec{B}) \right] \\ &\quad - \gamma^2 \left[\vec{A}_\parallel \cdot \vec{B}_\parallel - \cancel{A_0 (\vec{\beta} \cdot \vec{B})} - \cancel{B_0 (\vec{\beta} \cdot \vec{A})} + \beta^2 A_0 B_0 \right] \\ &\quad - \vec{A}_\perp \cdot \vec{B}_\perp. \end{aligned}$$

Regrouping the terms,

$$A'_0 B'_0 - \vec{A}' \cdot \vec{B}' = \cancel{\gamma^2 (1 - \beta^2)} A_0 B_0 + \gamma^2 \left[(\vec{\beta} \cdot \vec{A})(\vec{\beta} \cdot \vec{B}) - \vec{A}_\parallel \cdot \vec{B}_\parallel \right] - \vec{A}_\perp \cdot \vec{B}_\perp.$$

since

$$\gamma^2 (1 - \beta^2) = 1.$$

We also know that

$$\vec{A}_\parallel = \frac{\vec{\beta} \cdot \vec{A}}{\beta^2} \vec{\beta}, \quad \vec{B}_\parallel = \frac{\vec{\beta} \cdot \vec{B}}{\beta^2} \vec{\beta},$$

such that

$$\vec{A}_\parallel \cdot \vec{B}_\parallel = \frac{(\vec{\beta} \cdot \vec{A})(\vec{\beta} \cdot \vec{B})}{\beta^2}.$$

Therefore,

$$A'_0 B'_0 - \vec{A}' \cdot \vec{B}' = A_0 B_0 + \gamma^2 \left[(\vec{\beta} \cdot \vec{A})(\vec{\beta} \cdot \vec{B}) - \frac{(\vec{\beta} \cdot \vec{A})(\vec{\beta} \cdot \vec{B})}{\beta^2} \right] - \vec{A}_\perp \cdot \vec{B}_\perp \Rightarrow$$

$$A'_0 B'_0 - \vec{A}' \cdot \vec{B}' = A_0 B_0 - \cancel{\gamma^2 (1 - \beta^2)} \left[\frac{(\vec{\beta} \cdot \vec{A})(\vec{\beta} \cdot \vec{B})}{\beta^2} \right] - \vec{A}_\perp \cdot \vec{B}_\perp \Rightarrow$$

$$A'_0 B'_0 - \vec{A}' \cdot \vec{B}' = A_0 B_0 - \vec{A}_\parallel \cdot \vec{B}_\parallel - \vec{A}_\perp \cdot \vec{B}_\perp.$$

At this point, using

$$\vec{A} \cdot \vec{B} = \vec{A}_\parallel \cdot \vec{B}_\parallel + \vec{A}_\perp \cdot \vec{B}_\perp,$$

one obtains

$$\boxed{A'_0 B'_0 - \vec{A}' \cdot \vec{B}' = A_0 B_0 - \vec{A} \cdot \vec{B}}.$$

Thus,

$$\boxed{A' \cdot B' = A \cdot B}.$$

C) Length, proper time and time dilation

Consider a particle at rest at the origin of the spatial coordinate system. The world line of a light ray satisfies

$$x^2 + y^2 + z^2 = c^2 t^2.$$

We can draw a diagram distinguishing events inside, outside, and on the light cone.

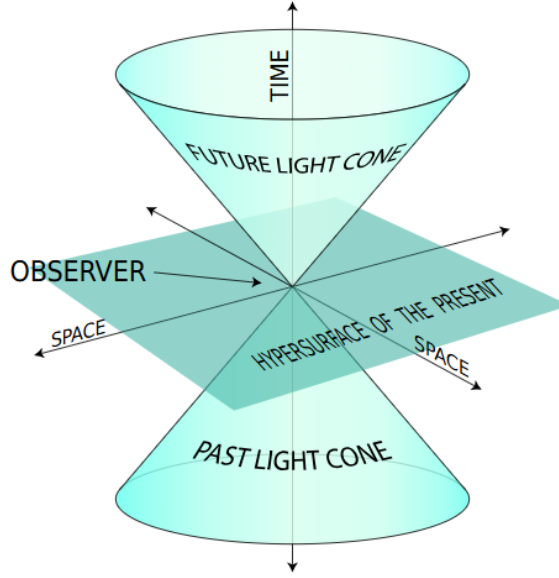


Figure 7: The light cone.

Consider two events $P_1(t_1, \vec{x}_1)$ and $P_2(t_2, \vec{x}_2)$, the spacetime interval between them is invariant:

$$s_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2. \quad (25)$$

There are three possibilities:

1. **Time-like interval:** $s_{12}^2 > 0$;

There exists a reference frame K' in which $\vec{x}'_1 = \vec{x}'_2$, and therefore $s_{12}^2 = c^2(t'_1 - t'_2)^2 > 0$. The two events occur at the same spatial position in K' , but at different times.

2. **Space-like interval:** $s_{12}^2 < 0$;

There exists a reference frame K' in which $t'_1 = t'_2$, and hence $s_{12}^2 = -|\vec{x}'_1 - \vec{x}'_2|^2 < 0$. The two events are simultaneous in K' but are separated in space.

3. **Light-like interval:** $s_{12}^2 = 0$;

The events can be connected by a light signal.

Proper time:

Consider a particle with velocity $\vec{u}(t)$ in the reference frame K .

Since

$$d\vec{x} = \vec{u} dt,$$

Eq. (25) gives the following

$$ds^2 = c^2 dt^2 - |d\vec{x}|^2 = c^2 dt^2 (1 - \beta^2),$$

where $\beta = u/c$.

In the reference frame K' in which the particle is instantaneously at rest (proper),

$$\vec{u}'(t') = 0, \quad d\vec{x}' = 0, \quad dt' \equiv d\tau.$$

Since the interval is invariant,

$$ds = c d\tau \Rightarrow c^2 d\tau^2 = c^2 dt^2 (1 - \beta^2).$$

Hence,

$$d\tau = dt \sqrt{1 - \beta^2(t)} = \frac{dt}{\gamma(t)}. \quad (26)$$

The quantity τ is the proper time of the particle.

Consequently,

$$dt = \frac{d\tau}{\sqrt{1 - \beta^2}},$$

which is the time dilation of the particle. By integrating (26), one gets:

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1 - \beta^2(\tau)}} = \int_{\tau_1}^{\tau_2} \gamma(\tau) d\tau. \quad (27)$$

Examples**1- Muon decay:**

Pions ($\pi^\pm$) decay to produce muons ($\mu^\pm$). The mean lifetime of the muon is written as approximately

$$\tau_0 \simeq 2.2 \times 10^{-6} \text{ s}.$$

At light speed, secondary cosmic-ray muons should decay after $c\tau_0 \simeq 660 \text{ m}$. However, their dilated lifetime is $\gamma\tau_0$ and they survive all the way down through the atmosphere. Suppose that they were produced at a height of 33 km. Thus,

$$\gamma c\tau_0 = 33 \text{ km} \Rightarrow \gamma = 50 \Rightarrow \beta \simeq 0.9996.$$

2- Fermilab pions:

At Fermilab charged pions with an energy of approximately 200 GeV are produced and transported 300 m with less than 3% loss. With a lifetime of approximately

$$\tau_0 \simeq 2.56 \times 10^{-8} \text{ s},$$

the Galilean decay distance should be $c\tau_0 \simeq 7.7 \text{ m}$ and only $e^{-300/7.7} \simeq 10^{-17}$ should survive. However, at

$$E_\pi = 200 \text{ GeV} \Rightarrow \gamma = 1400 \Rightarrow \gamma c\tau_0 = 11 \text{ km}.$$

3- Atomic clocks:

Four Cs atomic clocks flown around the world twice in aircrafts and were compared to a stationary one at US Naval Observatory [11]. The differences were approximately:

$$\begin{array}{lll} -59 \pm 10 \text{ ns}, & -40 \pm 23 \text{ ns}, & \text{eastward} \\ 273 \pm 7 \text{ ns}, & 275 \pm 21 \text{ ns}, & \text{westward.} \end{array}$$

D) Relativistic Doppler Effect

The phase of a wave (crest counting) is invariant under a Lorentz transformation. Consider

$$\phi = \omega t - \vec{k} \cdot \vec{x} = \omega' t' - \vec{k}' \cdot \vec{x}'. \quad (28)$$

Or, equivalently,

$$c\phi = \omega ct - c\vec{k} \cdot \vec{x} = \omega' ct' - c\vec{k}' \cdot \vec{x}'.$$

Let us define $\vec{\beta} = (v/c)\hat{z}$, and the wave vectors $\vec{k} = k\hat{z}$, and $\vec{k}' = k'\hat{z}'$.

Using the inverse Lorentz transformation,

$$ct = \gamma(ct' + \beta z') \quad \text{and} \quad z = \gamma(z' + \beta ct'),$$

one obtains the transformation law for the wave four-vector,

$$\begin{aligned} c\phi &= \omega ct - ckz = \\ &= \omega\gamma(ct' + \beta z') - ck\gamma(z' + \beta ct') = \\ &= \gamma(\omega - \beta ck)ct' - \gamma(ck - \beta\omega)z' \equiv \\ &= \omega' ct' - ck' z', \end{aligned}$$

where

$$\begin{cases} \omega' = \gamma(\omega - \beta ck), \\ ck' = \gamma(ck - \beta\omega). \end{cases}$$

Now, defining $\omega = ck_0$ and $\omega' = ck'_0$, we have the wave four-vector:

$$K = (ck_0, c\vec{k}),$$

which transforms according to Lorentz transformation.

In general, for a wave vector $\vec{k}$ pointing to an arbitrary direction (but still keeping the boost $\vec{\beta} = \beta\hat{z}$), we must replace these equations by:

$$\begin{cases} \omega' = \gamma(\omega - \beta ck_z), \\ ck'_z = \gamma(ck_z - \beta\omega), \end{cases}$$

and the (unchanged) components perpendicular to the boost are:

$$ck'_x = ck_x, \quad ck'_y = ck_y.$$

Generalizing the directions:

$$\begin{cases} k'_0 = \gamma(k_0 - \vec{\beta} \cdot \vec{k}), \\ k'_{\parallel} = \gamma(k_{\parallel} - \beta k_0), \\ \vec{k}'_{\perp} = \vec{k}_{\perp}. \end{cases} \quad (29)$$

Light waves

For light plane waves, the magnitude of the wave vector is related to the frequency by

$$|\vec{k}| = \frac{\omega}{c},$$

such that the phase of a plane wave is invariant:

$$c^2 k_0^2 - c^2 |\vec{k}|^2 = \frac{\omega^2}{c^2} - c^2 |\vec{k}|^2 = 0.$$

$$c^2 k_0'^2 - c^2 |\vec{k}'|^2 = \frac{\omega'^2}{c^2} - c^2 |\vec{k}'|^2 = 0.$$

Thus, the frequency-wave number four-vector,

$$K = (ck_0, c\vec{k}),$$

has null modulus:

$$K^2 = K \cdot K = c^2 k_0^2 - c^2 |\vec{k}|^2 = 0.$$

So does the spacetime four-vector,

$$X = (ct, \vec{x}),$$

since

$$X^2 = X \cdot X = c^2 t^2 - |\vec{x}|^2 = 0,$$

is the equation of a light pulse.

They are four-vectors of a special case whose squared modulus is null.

Now, their scalar product

$$K \cdot X = \Phi$$

is the invariant phase, and

$$\begin{aligned} \Phi^2 &= (K \cdot X)^2 = (k_0 ct - c\vec{k} \cdot \vec{x})^2 = \omega^2 t^2 - c^2 (\vec{k} \cdot \vec{x})^2 \\ &= c^2 (\omega t + \vec{k} \cdot \vec{x})(\omega t - \vec{k} \cdot \vec{x}) = \\ &\equiv c^2 \Phi^+ \Phi^-, \end{aligned}$$

where each factorization represents, respectively, a wave propagating with phase velocity $v_f < 0$ and $v_f > 0$.

Now, returning to (29) and making an angle θ between the direction of the wave and the boost velocity. For the wave vector, we have

$$k_{\parallel} = k \cos \theta,$$

and the two perpendicular components

$$k_{1\perp}, \quad k_{2\perp}.$$

The Lorentz transformations give

$$\omega' = \gamma(\omega - \beta ck \cos \theta).$$

Since

$$ck = \omega,$$

we obtain

$$\boxed{\omega' = \gamma\omega(1 - \beta \cos \theta)}, \quad (29)$$

which is the relativistic (transverse) Doppler shift.

For the direction of propagation,

$$\tan \theta' = \frac{k'_\perp}{k'_\parallel}.$$

Since

$$k'_\perp = k_\perp = k \sin \theta,$$

and

$$k'_\parallel = \gamma \left(k_\parallel - \beta \frac{\omega}{c} \right),$$

we have

$$\tan \theta' = \frac{k \sin \theta}{\gamma (k \cos \theta - \beta \omega/c)}.$$

Using

$$\omega = ck,$$

we finally obtain

$$\boxed{\tan \theta' = \frac{\sin \theta}{\gamma(\cos \theta - \beta)}} \quad (30)$$

which gives the relativistic aberration of light (lighthouse effect).

4. Velocity addition, four-vector formulation

Consider the following situation:

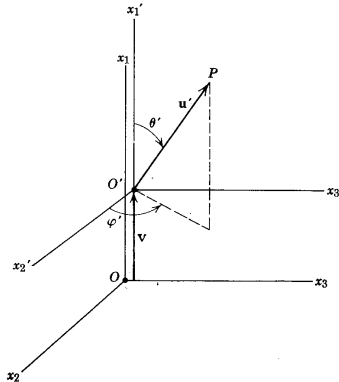


Figure 8: Addition of velocities.

We have

$$\begin{aligned} dx_0 &= \gamma_v (dx'_0 + \beta dx'_1), \\ dx_1 &= \gamma_v (dx'_1 + \beta dx'_0), \\ dx_2 &= dx'_2, \\ dx_3 &= dx'_3, \end{aligned}$$

where

$$\gamma_v = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

and, in order to determine the corresponding Lorentz factors,

$$\gamma_u = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}, \quad \gamma_{u'} = \left(1 - \frac{u'^2}{c^2}\right)^{-1/2}.$$

For the first component of the velocity,

$$u_1 = \frac{dx_1}{dt} = \frac{dx_1}{dx_0/c} = \frac{\gamma_v (dx'_1 + \beta dx'_0)}{\gamma_v (dx'_0 + \beta dx'_1)/c} = \frac{u'_1 + v}{1 + \frac{vu'_1}{c^2}}.$$

Similarly,

$$u_2 = \frac{dx_2}{dt} = \frac{dx'_2}{\gamma_v (dx'_0 + \beta dx'_1)/c} = \frac{u'_2}{\gamma_v \left(1 + \frac{vu'_1}{c^2}\right)},$$

and

$$u_3 = \frac{u'_3}{\gamma_v \left(1 + \frac{vu'_1}{c^2}\right)}.$$

Thus, separating the components parallel and perpendicular to $\vec{v}$,

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + \frac{\vec{v} \cdot \vec{u}'}{c^2}}, \quad \vec{u}_{\perp} = \frac{\vec{u}'_{\perp}}{\gamma_v \left(1 + \frac{\vec{v} \cdot \vec{u}'}{c^2}\right)}. \quad (31)$$

The angle θ is therefore given by

$$\tan \theta = \frac{u_{\perp}}{u_{\parallel}} = \frac{\frac{u' \sin \theta'}{\gamma_v \left(1 + \frac{vu' \cos \theta'}{c^2}\right)}}{\frac{u' \cos \theta' + v}{1 + \frac{vu' \cos \theta'}{c^2}}} = \frac{u' \sin \theta'}{\gamma_v (u' \cos \theta' + v)}. \quad (32')$$

For the magnitude of the velocity,

$$\begin{aligned}
 u = \sqrt{u_{\parallel}^2 + u_{\perp}^2} &= \sqrt{\frac{(u' \cos \theta' + v)^2}{\left(1 + \frac{vu' \cos \theta'}{c^2}\right)^2} + \frac{(u' \sin \theta')^2}{\gamma_v^2 \left(1 + \frac{vu' \cos \theta'}{c^2}\right)^2}} = \\
 &= \frac{\sqrt{u'^2 \cos^2 \theta' + 2u'v \cos \theta' + v^2 + u'^2 \sin^2 \theta' \left(1 - \frac{v^2}{c^2}\right)}}{1 + \frac{vu' \cos \theta'}{c^2}} = \\
 &= \frac{\sqrt{u'^2 + v^2 + 2u'v \cos \theta' - \frac{u'^2 v^2 \sin^2 \theta'}{c^2}}}{1 + \frac{u'v \cos \theta'}{c^2}}. \tag{32''}
 \end{aligned}$$

And, analogously, for $\vec{u}'$ by replacing $\vec{v} \rightarrow -\vec{v}$.

For $v \ll c$, Eqs. (31) and (32) reduce to the Galilean transformation $u = u' + v$.

In the case in which $\vec{u}$, $\vec{u}'$ and $\vec{v}$ are parallel:

$$\boxed{u = \frac{u' + v}{1 + \frac{vu'}{c^2}}}. \tag{33}$$

If $u' = c \Rightarrow u = c$, which shows that there are no velocities greater than that of light.

From Eq. (31), we see that $\vec{u}$ does not transform as a four-vector.

Note that

$$\begin{aligned}
 \gamma_u &= \left(1 - \frac{u^2}{c^2}\right)^{-1/2} = \left(1 - \frac{u_{\parallel}^2 + u_{\perp}^2}{c^2}\right)^{-1/2} = \\
 &= \left[1 - \frac{(u'_{\parallel} + v)^2}{\left(1 + \frac{\vec{v} \cdot \vec{u}'}{c^2}\right)^2 c^2} - \frac{u_{\perp}'^2}{\gamma_v^2 \left(1 + \frac{\vec{v} \cdot \vec{u}'}{c^2}\right)^2 c^2}\right]^{-1/2} = \\
 &= \left[\gamma_v^2 \left(1 + \frac{\vec{v} \cdot \vec{u}'}{c^2}\right)^2 - \gamma_v^2 \frac{(u'_{\parallel} + v)^2}{c^2} - \frac{u_{\perp}'^2}{c^2}\right]^{-1/2} \gamma_v \left(1 + \frac{\vec{v} \cdot \vec{u}'}{c^2}\right). \tag{34}
 \end{aligned}$$

Developing the term inside the square brackets,

$$\begin{aligned}
 &\gamma_v^2 + 2\cancel{\gamma_v^2 \frac{vu'_{\parallel}}{c^2}} + \gamma_v^2 \frac{v^2}{c^2} - \gamma_v^2 \frac{u_{\parallel}'^2}{c^2} - 2\cancel{\gamma_v^2 \frac{u_{\parallel}' v}{c^2}} - \gamma_v^2 \frac{v^2}{c^2} - \frac{u_{\perp}'^2}{c^2} = \\
 &= \cancel{\gamma_v^2 \left(1 - \frac{v^2}{c^2}\right)^1} + \gamma_v^2 \frac{u_{\parallel}'^2}{c^2} \left(\frac{v^2}{c^2} - 1\right)^{-1/\gamma_v^2} - \frac{u_{\perp}'^2}{c^2} = \\
 &= 1 - \frac{u_{\parallel}'^2}{c^2} - \frac{u_{\perp}'^2}{c^2} = 1 - \frac{u'^2}{c^2} = \frac{1}{\gamma_{u'}^2}.
 \end{aligned}$$

Therefore,

$$\boxed{\gamma_u = \gamma_{u'} \gamma_v \left(1 + \frac{\vec{v} \cdot \vec{u}'}{c^2} \right)}. \quad (34)$$

Substituting this result into Eq. (31),

$$\gamma_u u_{\parallel} = \gamma_v (\gamma_{u'} u'_{\parallel} + v \gamma_{u'}) , \quad \gamma_u \vec{u}_{\perp} = \gamma_{u'} \vec{u}'_{\perp}. \quad (35)$$

This transforms under a Lorentz transformation if we define the velocity four-vector

$$(U_0, \vec{U}) \equiv (\gamma_u c, \gamma_u \vec{u}), \quad (35)$$

which is obtained from the space-time four-vector by differentiation with respect to proper time:

$$U \equiv \frac{dX}{d\tau}. \quad (36)$$

Thus,

$$U_0 = \frac{dx_0}{d\tau} = \frac{dx_0}{dt} \frac{dt}{d\tau} = c \frac{dt}{d\tau} = \gamma_u c, \quad (36')$$

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt} \frac{dt}{d\tau} = \vec{u} \gamma_u = \gamma_u \vec{u}. \quad (36'')$$

5. Relativistic momentum and energy of a particle

For low velocities ($v \ll c$),

$$\vec{p} = m\vec{u}, \quad E = E(0) + \frac{1}{2}mu^2, \quad (37)$$

where m is the mass, $\vec{u}$ is the velocity, and $E(0)$ is the rest energy of the particle (which can be disregarded if $v \ll c$).

A possible generalization (of the first postulate) is

$$\vec{p} = \mathcal{M}(u)\vec{u}, \quad E = \mathcal{E}(u), \quad (38)$$

with the limiting values

$$\mathcal{M}(0) = m, \quad \left. \frac{\partial^2 \mathcal{E}}{\partial u^2} \right|_{u=0} = \frac{m}{2}, \quad (39)$$

and with $\mathcal{M}(u)$ and $\mathcal{E}(u)$ being monotonic functions and well behaved in their arguments.

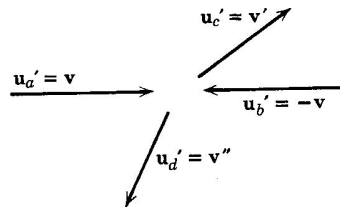


Figure 9: The collision of 2 identical particles in the center-of-mass frame K' .

Suppose an elastic collision of two identical particles is observed in the reference frame K' (center-of-mass frame), as given in Fig. 9.

The conservation laws for $(\vec{p}, E)$ are

$$\vec{p}'_a + \vec{p}'_b = \vec{p}'_c + \vec{p}'_d \Rightarrow \mathcal{M}(v)\vec{v} - \mathcal{M}(v)\vec{v} = \mathcal{M}(v')\vec{v}' + \mathcal{M}(v'')\vec{v}'', \quad (40)$$

$$E'_a + E'_b = E'_c + E'_d \Rightarrow \mathcal{E}(v) + \mathcal{E}(v) = \mathcal{E}(v') + \mathcal{E}(v''). \quad (40')$$

For identical particles, $\mathcal{E}(v') = \mathcal{E}(v'') \Rightarrow v' = v'' = v$, for monotonic functions. The first equation therefore becomes

$$0 = \mathcal{M}(v')\vec{v}' + \mathcal{M}(v')\vec{v}'' \Rightarrow \vec{v}' = -\vec{v}''.$$

Now let K be moving with velocity $-\vec{v} = -v\hat{z}$ with respect to K' :

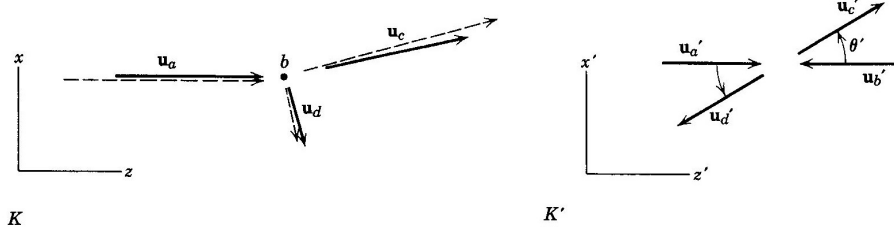


Figure 10: The collision of 2 identical particles in the laboratory frame K and in the center-of-mass frame K' .

From Eq. (31),

$$\vec{u}_a = \frac{2\vec{v}}{1 + v^2/c^2} = \frac{2c\vec{\beta}}{1 + \beta^2}, \quad \vec{\beta} = \frac{\vec{v}}{c}, \quad (41)$$

and the components of the velocities of the outgoing particles are

$$\begin{aligned} (u_c)_x &= \frac{c\beta \sin \theta'}{\gamma(1 + \beta^2 \cos \theta')}, & (u_c)_z &= \frac{c\beta(1 + \cos \theta')}{1 + \beta^2 \cos \theta'}, \\ (u_d)_x &= \frac{c\beta \sin \theta'}{\gamma(1 - \beta^2 \cos \theta')}, & (u_d)_z &= \frac{c\beta(1 - \cos \theta')}{1 - \beta^2 \cos \theta'}, \end{aligned} \quad (42)$$

where

$$\gamma = (1 - \beta^2)^{-1/2}.$$

In the reference frame K we then have, for momentum conservation,

$$\mathcal{M}(u_a)\vec{u}_a + \mathcal{M}(u_b)\vec{u}_b = \mathcal{M}(u_c)\vec{u}_c + \mathcal{M}(u_d)\vec{u}_d, \quad (43)$$

and, for energy conservation,

$$\mathcal{E}(u_a) + \mathcal{E}(u_b) = \mathcal{E}(u_c) + \mathcal{E}(u_d), \quad (43')$$

in which only $u_b = 0$.

Consider a skimming collision ($\theta' \rightarrow 0$), then $\vec{u}_c \simeq \vec{u}_a, \vec{u}_d \rightarrow 0$. The x component of the momentum conservation equation gives

$$\begin{aligned} 0 &= \mathcal{M}(u_c) \frac{c\beta \sin \theta'}{\gamma(1 + \beta^2 \cos \theta')} - \mathcal{M}(u_d) \frac{c\beta \sin \theta'}{\gamma(1 - \beta^2 \cos \theta')} \Rightarrow \\ &\Rightarrow \mathcal{M}(u_c) = \left(\frac{1 + \beta^2 \cos \theta'}{1 - \beta^2 \cos \theta'} \right) \mathcal{M}(u_d). \end{aligned}$$

This relation is valid for every θ' and, in particular, for $\theta' \rightarrow 0$.
In the limit $\theta' \rightarrow 0$, $\vec{u}_c = \vec{u}_a$, $\vec{u}_d = 0$, and therefore

$$\mathcal{M}(u_a) = \left(\frac{1 + \beta^2}{1 - \beta^2} \right) \mathcal{M}(0). \quad (44)$$

From Eq. (41),

$$\gamma_a = \frac{1}{\sqrt{1 - u_a^2/c^2}} = \frac{1}{\sqrt{1 - \left[\frac{2\beta}{1+\beta^2} \right]^2}} = \frac{1 + \beta^2}{1 - \beta^2}. \quad (45)$$

Therefore,

$$\mathcal{M}(u_a) = \gamma_a \mathcal{M}(0) = \gamma_a m,$$

and consequently

$$\boxed{\vec{p} = \gamma m \vec{u} = \frac{m \vec{u}}{\sqrt{1 - u^2/c^2}}}. \quad (46)$$

For θ' small, Eq. (43) gives

$$\mathcal{E}(u_a) + \mathcal{E}(0) = \mathcal{E}(u_c) + \mathcal{E}(u_d). \quad (47)$$

Assuming that the factor γ can be expanded in a binomial series,

$$\gamma \simeq 1 + \frac{1}{2} \frac{u^2}{c^2} + \dots,$$

we have

$$\lim_{v \rightarrow 0} \mathcal{E}(u) = \mathcal{E}(0) + \frac{1}{2} m u^2 = \mathcal{E}(0) \left[1 + \frac{1}{2} \frac{m u^2}{\mathcal{E}(0)} \right] = \mathcal{E}(0) \gamma(u), \quad \text{if } \boxed{\mathcal{E}(0) = m c^2}. \quad (50)$$

The kinetic energy is therefore

$$T(u) = \mathcal{E}(u) - \mathcal{E}(0) = m c^2 [\gamma(u) - 1] = m c^2 \left[\frac{1}{\sqrt{1 - u^2/c^2}} - 1 \right] \Rightarrow \boxed{T(u) = m c^2 (\gamma - 1)}. \quad (49)$$

But also,

$$T(u) = \int \vec{F} \cdot d\vec{x} = \int \frac{d}{dt} (\gamma m \vec{u}) \cdot d\vec{x} = \int m d(\gamma \vec{u}) \cdot \frac{d\vec{x}}{dt} = m \int_0^u u d(\gamma u).$$

Now,

$$\frac{d(\gamma u)}{du} = \frac{d}{du} \left[\left(1 - \frac{u^2}{c^2} \right)^{-1/2} u \right] = \frac{1}{(1 - u^2/c^2)^{3/2}} = \gamma^3.$$

Hence,

$$T(u) = \mathcal{E}(u) - \mathcal{E}(0) = \int_{\mathcal{E}(0)}^{\mathcal{E}(u)} dE = m \int_0^u \gamma^3 u du = \frac{m}{2} \int_0^{u^2} \gamma^3 d(u^2).$$

Equivalently,

$$\frac{d\mathcal{E}}{d(u^2)} = \frac{m}{2}\gamma^3 = \frac{m}{2(1 - u^2/c^2)^{3/2}}.$$

Integrating, with $z \equiv 1 - u^2/c^2$:

$$\int_0^u \frac{u \, du}{(1 - u^2/c^2)^{3/2}} = \frac{c^2}{2} \int_1^{1-u^2/c^2} \frac{-dz}{z^{3/2}} = \frac{c^2}{\sqrt{z}} \Big|_1^{1-u^2/c^2} = \frac{c^2}{\sqrt{1 - u^2/c^2}} - c^2.$$

Therefore,

$$\mathcal{E}(u) - \mathcal{E}(0) = \frac{mc^2}{\sqrt{1 - u^2/c^2}} - mc^2. \quad (48)$$

Thus, consistently with Eq. (50), $\mathcal{E}(0) = mc^2$, we have

$$\boxed{E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}}}. \quad (51)$$

The momentum–energy four-vector is

$$P \equiv mU = (m\gamma c, m\gamma \vec{u}) = \left(\frac{E}{c}, \vec{p} \right).$$

Its components are conserved in the same way in a collision:

$$\begin{aligned} \text{energy:} \quad & \sum_a (p_0)_a^i - \sum_b (p_0)_b^f = \Delta_0 = 0, \\ \text{momentum:} \quad & \sum_a (\vec{p})_a^i - \sum_b (\vec{p})_b^f = \vec{\Delta}_0 = \vec{0}. \end{aligned} \quad (52)$$

The invariant modulus of this four-vector is

$$P^2 = m^2 \gamma^2 (c^2 - u^2) = m^2 \gamma^2 c^2 \left(1 - \frac{u^2}{c^2} \right) = m^2 c^2. \quad (54)$$

Consequently,

$$\frac{E^2}{c^2} - p^2 = m^2 c^2,$$

and hence

$$\boxed{E = \sqrt{p^2 c^2 + m^2 c^4}}. \quad (55)$$

Also, comparing Eq. (46) with Eq. (51),

$$\vec{p} = m\gamma \vec{u} = \frac{E}{c^2} \vec{u},$$

so that

$$\vec{u} = \frac{c^2 \vec{p}}{E}. \quad (53)$$

Suppose a particle in K has momentum

$$\vec{p} = \vec{p}_\perp + p_\parallel \hat{\mathbf{z}}.$$

There exists a reference frame K' in which $p'_\parallel = 0$.

Then

$$\vec{p}' = \vec{p}'_\perp = \vec{p}_\perp, \quad \frac{E'}{c} = \sqrt{p_\perp^2 + m^2 c^2} \equiv \Omega. \quad (56)$$

If ζ is the rapidity of the Lorentz transformation ($K \leftrightarrow K'$), then, in K (inverting Eq. (21)),

$$p_\perp, \quad p_\parallel = \Omega \sinh \zeta, \quad \frac{E}{c} = \Omega \cosh \zeta. \quad (57)$$

The quantity Ω/c is called the *transverse mass* (associated with $p_\perp$).

If, in K' , $\vec{p}'_\perp = 0$, then

$$\vec{p}' = 0 \quad (\text{rest frame}),$$

and

$$p = p_\parallel = mc \sinh \zeta, \quad E = mc^2 \cosh \zeta. \quad (58)$$

6. Mathematical properties of spacetime

In classical and quantum mechanics, three-dimensional rotations can be described as a group of coordinate transformations that preserve the norm of the vector $\vec{x}$, called $\text{SO}(3)$. In special relativity, Lorentz transformations of the four-dimensional coordinates $(x^0, \vec{x})$ preserve the quantity

$$s^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2. \quad (59)$$

Therefore, the group of transformations that keeps s^2 invariant is the homogeneous Lorentz group.

The group of transformations that preserves

$$s^2(x, y) = (x^0 - y^0)^2 - (x^1 - y^1)^2 - (x^2 - y^2)^2 - (x^3 - y^3)^2$$

is the inhomogeneous Lorentz group, or the Poincaré group.

The mathematical equations expressing the laws of nature must be covariant, that is, invariant in form under transformations of the Lorentz group.

Consider a tensor analysis in a four-dimensional vector space: spacetime (continuum), defined in a four-dimensional space with coordinates

$$x^0, x^1, x^2, x^3.$$

The transformed coordinates

$$x'^0, x'^1, x'^2, x'^3$$

are obtained by a well-defined rule:

$$x'^\alpha = x'^\alpha(x^0, x^1, x^2, x^3), \quad (\alpha = 0, 1, 2, 3). \quad (60)$$

A tensor of class k associated with the point in spacetime x is defined by its properties under the transformation $x \rightarrow x'$.

A scalar (tensor of $k = 0$) is simply an invariant quantity:

$$s^2 \quad \text{is a Lorentz scalar.}$$

Vectors (tensors of $k = 1$), with four components, are divided into

$$A^\alpha = (A^0, A^1, A^2, A^3), \quad \text{contravariant,}$$

and

$$B_\alpha = (B_0, B_1, B_2, B_3), \quad \text{covariant.}$$

The contravariant vectors transform according to

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta = \sum_{\beta=0}^3 \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta, \quad (61)$$

where summation over repeated indices is understood.

The covariant vectors transform according to

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta = \sum_{\beta=0}^3 \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta, \quad (62)$$

where summation over repeated indices is understood.

The partial derivatives are obtained from the inverse transformation of Eq. (60).

Tensors of rank 2 ($k = 2$) consist of 16 quantities (components):

- contravariant ($F^{\alpha\beta}$):

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}; \quad (63)$$

- covariant ($G_{\alpha\beta}$):

$$G'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} G_{\gamma\delta}; \quad (64)$$

- mixed (H^α_β):

$$H'^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x^\delta}{\partial x'^\beta} H^\gamma_\delta. \quad (65)$$

The scalar (invariant) product of two vectors is defined by

$$B \cdot A \equiv B_\alpha A^\alpha. \quad (66)$$

It is invariant:

$$B'_\alpha A'^\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta \frac{\partial x'^\alpha}{\partial x^\gamma} A^\gamma = \frac{\partial x^\beta}{\partial x^\gamma} B_\beta A^\gamma = \delta^\beta_\gamma B_\beta A^\gamma = B_\beta A^\beta = B \cdot A.$$

In differential form, the infinitesimal interval that defines the geometry of space is

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \quad (67)$$

The metric is a special case of the differential line element:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad (68)$$

where

$$g_{\alpha\beta} = g_{\beta\alpha}$$

is the metric tensor.

For flat spacetime of special relativity, g is diagonal:

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1. \quad (69)$$

The contravariant metric tensor is obtained from $g_{\alpha\beta}$, and for flat spacetime

$$g^{\alpha\beta} = g_{\alpha\beta}. \quad (70)$$

Note that the contraction gives

$$g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma. \quad (71)$$

Now, Eqs. (66) and (68) give

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = dx_\alpha dx^\alpha,$$

and therefore

$$x_\alpha = g_{\alpha\beta} x^\beta. \quad (72)$$

The inverse relation is

$$x^\alpha = g^{\alpha\beta} x_\beta. \quad (73)$$

Generalizing to $k \geq 2$,

$$\begin{aligned} F_{\dots\alpha\dots} &= g_{\alpha\beta} F^{\dots\beta\dots}, \\ G^{\dots\alpha\dots} &= g^{\alpha\beta} G_{\dots\beta\dots}. \end{aligned} \quad (74)$$

From Eq. (69), we have

$$A^\alpha = (A^0, \vec{A}), \quad A_\alpha = (A^0, -\vec{A}). \quad (75)$$

Therefore, the scalar product is

$$B \cdot A = B_\alpha A^\alpha = B_0 A^0 - \vec{B} \cdot \vec{A}.$$

Consider now the operators

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}, \quad \frac{\partial}{\partial x^\beta} = \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial}{\partial x'^\alpha}.$$

From Eqs. (62) and (61),

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta, \quad A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta.$$

Thus,

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right), \quad \partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right). \quad (76)$$

The four-divergence of the four-vector A is invariant:

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A}. \quad (77)$$

An analogous expression occurs in the Lorenz-gauge condition.

The d'Alembertian is the contraction

$$\square \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial (x^0)^2} - \nabla^2, \quad (78)$$

which is the operator appearing in the wave equation in vacuum:

$$\square \Psi = 0.$$

7. Matrix representation of Lorentz transformations

(Generators and infinitesimal transformations)

Consider the group of Lorentz transformations. The components of a contravariant four-vector can be written in the form of a column vector:

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \quad (79)$$

A scalar product between two four-vectors is defined by

$$(a, b) \equiv \tilde{a} b, \quad (80)$$

where $\tilde{a}$ denotes the transpose of a .

The metric tensor $g_{\alpha\beta}$ is represented by the 4×4 matrix

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (81)$$

with

$$g^2 = I,$$

where I is the 4×4 identity matrix.

The covariant vector is therefore

$$gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix} \equiv \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (82)$$

In this notation, the scalar product of two four-vectors is defined by

$$a \cdot b \equiv (a, gb) = (ga, b) = \tilde{a} g b. \quad (83)$$

Consider a group of linear coordinate transformations,

$$x' = Ax, \quad (84)$$

which preserve the norm:

$$x' \cdot x' = (x', gx') = \tilde{x}' gx' = \tilde{x} gx = (x, gx) = x \cdot x. \quad (85)$$

Substituting Eq. (84),

$$\begin{aligned} x' \cdot x' &= (Ax', gAx) = \\ &= \tilde{x}' \tilde{A} g Ax = \tilde{x} gx, \end{aligned}$$

and therefore

$$\tilde{A} g A = g. \quad (86)$$

Taking the determinant and using (86),

$$\det(\tilde{A} g A) = \det g (\det A)^2 = \det g.$$

Since

$$\det g = -1 \neq 0,$$

we obtain

$$\det A = \pm 1.$$

There are two classes of transformations:

1. proper Lorentz transformations:

$$\det A = +1;$$

2. improper Lorentz transformations:

$$\det A = -1.$$

The proper transformations contain the identity I and therefore form a group.

Improper transformations may have two signs, for example:

$$\text{spatial inversion:} \quad A = g \implies \det A = -1,$$

and

$$\text{space-time inversion:} \quad A = -I \implies \det A = +1.$$

Since A and g are 4×4 matrices, Eq. (86) represents $4^2 = 16$ equations, but only 10 are linearly independent. Hence, there are 6 free parameters.

The Lorentz group is therefore a group with six parameters:

1. three parameters specifying the relative orientation of the coordinate axes (Euler angles);
2. three parameters specifying the relative velocity of the two inertial systems (the components of $\vec{\beta}$).

For each set of six parameters A describing a proper Lorentz transformation, there is an inverse transformation. We consider only the proper transformations from now on. We make the ansatz

$$A = e^L, \quad (87)$$

where L is a 4×4 matrix, and

$$e^L = \sum_{k=0}^{\infty} \frac{L^k}{k!} = I + L + \frac{1}{2!}L^2 + \dots$$

Suppose that there is a similarity transformation, which does not alter the determinant, such that

$$L = SDS^{-1},$$

where D is diagonal. Then

$$\det L = \det(SDS^{-1}) = \det S \det D \det S^{-1} = \det D.$$

Thus, it is possible to diagonalize L .

Let the eigenvalues of D be on the diagonal:

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_4 \end{pmatrix}.$$

Then

$$\begin{aligned} e^D &= \sum_{k=0}^{\infty} \frac{D^k}{k!} = I + D + \frac{1}{2!}D^2 + \dots = \\ &= \begin{pmatrix} e^{\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_4} \end{pmatrix}. \end{aligned}$$

Therefore,

$$\det e^D = e^{\lambda_1} e^{\lambda_2} \dots = e^{\lambda_1 + \lambda_2 + \dots} = e^{\text{Tr } D}.$$

Hence,

$$\det A = \det e^L = e^{\text{Tr } L} = \pm 1.$$

For real L ,

$$\det A = -1$$

is excluded, and if

$$\text{Tr } L = 0,$$

then

$$\det A = +1,$$

as required for proper Lorentz transformations.

Thus, L is real and

$$\text{Tr } L = 0.$$

Now, from Eq. (86),

$$g(\tilde{A}gA)A^{-1} = ggA^{-1} \implies g\tilde{A}g = A^{-1}. \quad (88)$$

Using the definition (87),

$$\tilde{A} = e^{\tilde{L}}, \quad A^{-1} = e^{-L},$$

and

$$g\tilde{A}g = e^{g\tilde{L}g} = e^{g^2\tilde{L}} = e^{I\tilde{L}} = e^{\tilde{L}},$$

so that

$$g\tilde{L}g = -L. \quad (89')$$

Hence

$$g(g\tilde{L}g) = -gL = g^2\tilde{L}g = I\tilde{L}g = \tilde{L}g = \widetilde{Lg} \Rightarrow \widetilde{Lg} = -gL \quad (89'')$$

It follows that the matrix gL must be traceless and antisymmetric.

The matrix L can therefore be written in the general form

$$L = \begin{pmatrix} 0 & \ell_{01} & \ell_{02} & \ell_{03} \\ \ell_{01} & 0 & \ell_{12} & \ell_{13} \\ \ell_{02} & -\ell_{12} & 0 & \ell_{23} \\ \ell_{03} & -\ell_{13} & -\ell_{23} & 0 \end{pmatrix}. \quad (90)$$

The six fundamental generators of the group are the three generators of rotations in $\mathbb{R}^3$, S_1, S_2, S_3 , and the three generators of boosts, K_1, K_2, K_3 .

They may be represented by

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (91')$$

and

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (91'')$$

The matrices S_i generate rotations in space, whereas the matrices K_i generate boosts.

The squares of the generator matrices are

$$S_1^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_2^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (92')$$

and

$$K_1^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (92'')$$

It follows that

$$S_i^3 = -S_i, \quad K_i^3 = K_i,$$

where the first relation holds in the corresponding nontrivial spatial subspace and the second in the corresponding time-space subspace. The first suggests that they are pure complex numbers and the second that they are pure real numbers.

Thus, for a vector in the corresponding subspace,

$$(\hat{e}_i \cdot \vec{S})^3 = -\hat{e}_i \cdot \vec{S}, \quad (\hat{e}_i \cdot \vec{K})^3 = \hat{e}_i \cdot \vec{K}.$$

The matrices in Eq. (91) form a representation of the infinitesimal generators of the Lorentz group. They satisfy the commutation relations

$$\begin{aligned} [S_i, S_j] &= \epsilon_{ijk} S_k, \\ [S_i, K_j] &= \epsilon_{ijk} K_k, \\ [K_i, K_j] &= -\epsilon_{ijk} S_k. \end{aligned} \quad (99)$$

These relations specify the algebraic structure of the Lorentz group in $SL(2, \mathbb{C})$ or $O(3, 1)$. Let us now consider the decomposition of L :

$$L = -\vec{\omega} \cdot \vec{J} - \vec{\zeta} \cdot \vec{K}, \quad A = e^{-\vec{\omega} \cdot \vec{J} - \vec{\zeta} \cdot \vec{K}}. \quad (93)$$

The components of $\vec{\omega}$ and $\vec{\zeta}$ are the six parameters of the transformation. Consider the case $\vec{\omega} = 0$ and $\vec{\zeta} = \zeta \hat{e}_1 = \zeta \hat{x}$:

$$L = -\zeta K_1 = -\zeta K_X.$$

Then

$$A = e^L = e^{-\zeta K_1} = I - \zeta K_1 + \frac{\zeta^2}{2!} K_1^2 - \frac{\zeta^3}{3!} K_1^3 + \dots$$

and therefore, by (92):

$$\begin{aligned} A &= (I - K_1^2) - K_1 \left(\zeta + \frac{\zeta^3}{3!} + \dots \right) + K_1^2 \left(I + \frac{\zeta^2}{2!} + \dots \right) \\ &= (I - K_1^2) - K_1 \sinh \zeta + K_1^2 \cosh \zeta. \end{aligned} \quad (94)$$

Or, in matrix form,

$$A = \begin{pmatrix} \cosh \zeta & -\sinh \zeta & 0 & 0 \\ -\sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (95 = 21)$$

In the case $\zeta = 0$, $\vec{\omega} = \omega \hat{e}_3$. We have, similarly,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \omega & \sin \omega & 0 \\ 0 & -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (96)$$

or a rotation around $\hat{z}$ in the clockwise direction.

For a boost in an arbitrary direction, without rotation:

$$A = e^{-\vec{\zeta} \cdot \vec{K}}.$$

From

$$\beta = \tanh \zeta \quad \implies \quad \zeta = \tanh^{-1} \beta,$$

or

$$\vec{\zeta} = \hat{\beta} \tanh^{-1} \beta,$$

we obtain

$$A(\vec{\beta}) = e^{-\vec{\beta} \cdot \vec{K} \tanh^{-1} \beta}. \quad (97)$$

We leave it as an exercise to show that

$$A_{\text{boost}}(\vec{\beta}) = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1 + (\gamma - 1)\frac{\beta_1^2}{\beta^2} & (\gamma - 1)\frac{\beta_1\beta_2}{\beta^2} & (\gamma - 1)\frac{\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & (\gamma - 1)\frac{\beta_1\beta_2}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_2^2}{\beta^2} & (\gamma - 1)\frac{\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & (\gamma - 1)\frac{\beta_1\beta_3}{\beta^2} & (\gamma - 1)\frac{\beta_2\beta_3}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_3^2}{\beta^2} \end{pmatrix}. \quad (98)$$

which is, according to the explicit form,

$$x'^0 = \gamma \left(x^0 - \vec{\beta} \cdot \vec{x} \right),$$

and

$$\vec{x}' = \vec{x} + \left[\frac{(\gamma - 1)(\vec{\beta} \cdot \vec{x})\vec{\beta}}{\beta^2} - \gamma \vec{\beta} x^0 \right].$$

9. Invariance of electric charge; covariance of electromagnetism

Covariance of Maxwell's equations implies transformations of ρ , $\vec{J}$, $\vec{E}$, and $\vec{B}$. Consider the Lorentz force:

$$\frac{d\vec{p}}{dt} = q \left(\vec{E} + \frac{\vec{u} \times \vec{B}}{c} \right). \quad (124)$$

We have already seen that $\vec{p}$ transforms as a four-vector,

$$p^\alpha = (p^0, \vec{p}) = m(U^0, \vec{U}), \quad (55)$$

where

$$p^0 = \frac{E}{c}, \quad U^\alpha = \frac{dx^\alpha}{d\tau} = \gamma(c, \vec{u}), \quad (56)$$

with

$$d\tau = \frac{dt}{\gamma}. \quad (57)$$

Therefore,

$$\begin{aligned} \frac{d\vec{p}}{dt} &= \frac{d\vec{p}}{d\tau} \frac{d\tau}{dt} = \frac{1}{\gamma} \frac{d\vec{p}}{d\tau} \\ &= \frac{q}{\gamma} \left(\vec{E} + \frac{\vec{u} \times \vec{B}}{c} \right), \end{aligned} \quad (58)$$

and hence

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c} \left(U^0 \vec{E} + \vec{U} \times \vec{B} \right). \quad (125)$$

For the spatial part and the temporal part, we have

$$\frac{dp^0}{dt} = \frac{1}{\gamma} \frac{dp^0}{d\tau} = \frac{1}{\gamma} \frac{d(E/c)}{d\tau}, \quad (59)$$

where E is the relativistic energy. Its rate of change is

$$\frac{dE}{dt} = \int_V \vec{J} \cdot \vec{E} d^3x = \int_V \rho \vec{u} \cdot \vec{E} d^3x = q \vec{u} \cdot \vec{E}. \quad (60)$$

For a uniform current flux through the volume V ,

$$\frac{dp^0}{d\tau} = \frac{1}{\gamma c} q \vec{u} \cdot \vec{E} = \frac{q}{c} \vec{U} \cdot \vec{E}. \quad (126)$$

Thus, if $dp^\alpha/d\tau$ forms a four-vector, the right-hand side of Eq. (125), together with Eq. (126), must also form a four-vector.

Electric charge is invariant under Lorentz transformations (scalar Lorentz invariant), and J^μ is a four-vector. The temporal component of Eq. (126) is

$$\vec{E} \cdot \vec{u} = F^{0\beta} U_\beta. \quad (61)$$

For simplicity, consider Maxwell's equations in vacuum.

From the continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \quad (127)$$

Using Eqs. (76) and (77),

$$\partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A}, \quad \partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right). \quad (62)$$

We then define

$$J^\alpha = (c\rho, \vec{J}), \quad (128)$$

and the continuity equation becomes

$$\partial_\alpha J^\alpha = 0. \quad (129)$$

The charge is invariant under Lorentz transformations:

$$q = \int \rho d^3x = \int \rho' d^3x'. \quad (63)$$

The four-dimensional differential volume is invariant,

$$d^4x' = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} d^4x = d^4x, \quad (64)$$

and therefore

$$\rho d^3x = \rho' d^3x' \quad (65)$$

for the total charge.

In SI units, the equations for the potentials in the Lorenz gauge are

$$\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{J}, \quad (130)$$

and

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{\rho}{\epsilon_0}. \quad (66)$$

Since

$$\frac{1}{\epsilon_0 c} = \mu_0 c,$$

we may write

$$\frac{1}{c^2} \frac{\partial^2 (\phi/c)}{\partial t^2} - \nabla^2 (\phi/c) = \mu_0 J^0. \quad (131)$$

Thus, ϕ/c and $\vec{A}$ form a four-vector:

$$A^\alpha = \left(\frac{\phi}{c}, \vec{A} \right), \quad (132)$$

such that the four-dimensional wave equation and the Lorenz condition become

$$\begin{cases} \square A^\alpha = \partial_\beta \partial^\beta A^\alpha = \mu_0 J^\alpha, \\ \partial_\alpha A^\alpha = 0. \end{cases} \quad (133)$$

Expressing the fields in terms of the potentials,

$$\vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}. \quad (134)$$

Since

$$\phi = cA^0,$$

we have, for example,

$$\begin{aligned} E_x &= -\frac{\partial(cA^0)}{\partial x} - \frac{\partial A_x}{\partial t} \\ &= -c(\partial^1 A^0 - \partial^0 A^1), \end{aligned} \quad (135)$$

and

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2). \quad (67)$$

Since

$$\partial^\alpha = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right),$$

the six components $(\vec{E}, \vec{B})$ transform as the components of a second-rank antisymmetric tensor, the electromagnetic field tensor:

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha. \quad (136)$$

Explicitly,

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \quad (137)$$

The tensor with both indices covariant is

$$F_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}. \quad (138)$$

The dual tensor is obtained by the replacement

$$\vec{E} \rightarrow \vec{B}, \quad \vec{B} \rightarrow -\frac{\vec{E}}{c},$$

and is given by

$$\tilde{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}. \quad (140)$$

The four-index Levi-Civita tensor is defined by

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1, & \text{for an even permutation of } (0, 1, 2, 3), \\ -1, & \text{for an odd permutation,} \\ 0, & \text{if any two indices are equal.} \end{cases} \quad (139)$$

Maxwell's equations are

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}. \quad (68)$$

They can be written covariantly as

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta. \quad (141)$$

For $\beta = 0$,

$$\begin{aligned}\partial_\alpha F^{\alpha 0} &= \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} \\ &= \vec{\nabla} \cdot \vec{E}/c = \mu_0 J^0.\end{aligned}\quad (69)$$

For the spatial components, $\beta = i$,

$$\begin{aligned}\partial_\alpha F^{\alpha i} &= \partial_0 F^{0i} + \partial_j F^{ji} \\ &= -\frac{1}{c^2} \frac{\partial E_i}{\partial t} + (\vec{\nabla} \times \vec{B})_i \\ &= \mu_0 J^i.\end{aligned}\quad (70)$$

The two homogeneous Maxwell equations,

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad (71)$$

become

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0. \quad (142)$$

For the dual tensor,

$$\begin{aligned}\partial_\alpha \tilde{F}^{\alpha 0} &= \partial_i B_i = \vec{\nabla} \cdot \vec{B} = 0, \\ \partial_\alpha \tilde{F}^{\alpha i} &= \left[\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right]_i = 0.\end{aligned}\quad (72)$$

In terms of $F^{\alpha\beta}$, this can also be written as the Bianchi identity

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0. \quad (143)$$

The Lorentz-force equation is

$$\frac{dp^\alpha}{d\tau} = m \frac{dU^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta, \quad (144)$$

which is the covariant form of Eqs. (125) and (126).

For the macroscopic equations, introduce

$$G^{\alpha\beta} = G^{\alpha\beta}(\vec{D}, \vec{H}), \quad (73)$$

in analogy with

$$F^{\alpha\beta} = F^{\alpha\beta}(\vec{E}, \vec{B}),$$

that is,

$$\vec{E} \rightarrow \vec{D}, \quad \vec{B} \rightarrow \vec{H}, \quad c \rightarrow \frac{1}{\sqrt{\epsilon\mu}}.$$

Then,

$$\partial_\alpha G^{\alpha\beta} = \mu J^\beta, \quad \partial_\alpha \tilde{G}^{\alpha\beta} = 0. \quad (145)$$

Chapter 12 – Relativistic Particle Dynamics and Electromagnetic Fields

1. Lagrangian and Hamiltonian of a Relativistic Charged Particle in an External Electromagnetic Field

a) Equations of motion

The equations of motion may be written as

$$\frac{d\vec{p}}{dt} = e \left[\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right], \quad \frac{dE}{dt} = e \vec{u} \cdot \vec{E}. \quad (1)$$

They can be written in covariant form as

$$\frac{dp^\alpha}{d\tau} = \frac{e}{c} F^{\alpha\beta} u_\beta, \quad (2)$$

where m is the mass, τ is the proper time, and

$$u^\alpha = (\gamma c, \gamma \vec{u}) = \frac{p^\alpha}{m} \quad (74)$$

is the four-velocity of the particle.

Lagrangian formulation. The system is described by the generalized coordinates $q(t)$ and the generalized velocities $\dot{q}(t)$ through the Lagrangian function

$$L = L(q(t), \dot{q}(t), t).$$

The principle of minimum action states that the action

$$A = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt \quad (3)$$

is an extremum. Thus, $\delta A = 0$, and one obtains the Euler–Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0, \quad (4)$$

which are equivalent to Newton's laws.

b) Elementary treatment of the relativistic Lagrangian

The integral of the action must be a scalar Lorentz invariant, so that $\delta A = 0$ in all inertial reference frames. Substituting $t \rightarrow \gamma\tau$,

$$A = \int_{\tau_1}^{\tau_2} \mathcal{L} d\tau, \quad (5)$$

since $dt = \gamma d\tau$.

If $\delta A = 0$ is invariant, A is a scalar invariant. Since $d\tau$ is invariant, $\mathcal{L}$ must be a Lorentz scalar invariant, and consequently $\mathcal{L}$ is a scalar invariant.

A free particle (with no external fields) has constant four-velocity. The simplest invariant function is

$$u_\alpha u^\alpha = c^2, \quad (75)$$

and we may choose the Lagrangian

$$L_{\text{free}} = -mc^2 \sqrt{1 - \frac{u^2}{c^2}}. \quad (6)$$

Then

$$\frac{\partial L}{\partial q} = 0, \quad \frac{\partial L}{\partial u} = \frac{mu}{\sqrt{1 - u^2/c^2}} = c^2 m \gamma \frac{u}{c^2},$$

and, from Eq. (4),

$$\frac{d}{dt}(m\gamma\vec{u}) = 0,$$

which is Newton's law for a free particle in relativistic form.

Let a particle be immersed in a scalar potential Φ . The potential energy is

$$V = e\Phi.$$

In the nonrelativistic limit,

$$L = T - V \implies L_{\text{int}} = -e\Phi, \quad v \ll c. \quad (7)$$

We want to find the scalar invariant of the four-potential that reduces to $-e\Phi$ in the nonrelativistic limit. Since Φ is the temporal part of

$$A^\alpha = (\Phi, \vec{A}),$$

we can construct the scalar $u_\alpha A^\alpha$. Therefore, the simplest form of the interaction Lagrangian is

$$L_{\text{int}} = -\frac{e}{c} u_\alpha A^\alpha = -\frac{e}{c} (\gamma c, \gamma \vec{u}) (\Phi, \vec{A}) = -e\Phi + \frac{e}{c} \vec{u} \cdot \vec{A}. \quad (8)$$

Combining Eqs. (6) and (8),

$$L = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c} \vec{u} \cdot \vec{A} - e\Phi. \quad (9)$$

The canonical momentum $\vec{P}$ conjugate to the coordinate $\vec{x}$ is obtained from

$$P_i = \frac{\partial L}{\partial u_i} = \gamma m u_i + \frac{e}{c} A_i, \quad (10)$$

or, in vector notation,

$$\vec{P} = \vec{p} + \frac{e}{c}\vec{A}, \quad (11)$$

where $\vec{p} = \gamma m \vec{u}$ is the relativistic momentum of the particle.

The Hamiltonian is a function $H = H(\vec{x}, \vec{P})$; if it is independent explicitly of time, it is a constant. It is defined by

$$H = \vec{P} \cdot \vec{u} - L. \quad (12)$$

From Eq. (11),

$$\vec{p} = m\gamma\vec{u} = c\vec{P} - e\vec{A},$$

and therefore

$$\vec{u} = \frac{c\vec{P} - e\vec{A}}{m\gamma} = \frac{c\vec{P} - e\vec{A}}{\sqrt{(c\vec{P} - e\vec{A})^2 + m^2c^4}/c} = \frac{c(c\vec{P} - e\vec{A})}{\sqrt{(c\vec{P} - e\vec{A})^2 + m^2c^4}}. \quad (13)$$

Consequently,

$$\begin{aligned} H &= \vec{P} \cdot \vec{u} - L \\ &= \frac{c\vec{P} \cdot (c\vec{P} - e\vec{A})}{\sqrt{(c\vec{P} - e\vec{A})^2 + m^2c^4}} + \frac{mc^2}{\gamma} - \frac{e}{c}\vec{u} \cdot \vec{A} + e\Phi \\ &= \sqrt{(c\vec{P} - e\vec{A})^2 + m^2c^4} + e\Phi. \end{aligned} \quad (14)$$

Notice that

$$H = \sqrt{(c\vec{P} - e\vec{A})^2 + m^2c^4} + e\Phi \equiv W, \quad (76)$$

so that

$$(W - e\Phi)^2 - (c\vec{P} - e\vec{A})^2 = (mc^2)^2. \quad (15)$$

This is invariant and has the form of the scalar product

$$p^\alpha p_\alpha = (mc)^2. \quad (16)$$

Therefore,

$$p^\alpha = \left(\frac{E}{c}, \vec{p} \right) = \left(\frac{1}{c}(W - e\Phi), \vec{P} - \frac{e}{c}\vec{A} \right). \quad (17)$$

Covariant treatment of the relativistic Lagrangian

Covariantly, substituting x^α and u^α into the free-particle Lagrangian gives

$$\mathcal{L}_{\text{free}} = -mc\sqrt{u_\alpha u^\alpha}. \quad (18)$$

The action is therefore

$$A = -mc \int_{\tau_1}^{\tau_2} \sqrt{u_\alpha u^\alpha} d\tau, \quad (19)$$

with the normalization condition

$$u_\alpha u^\alpha = c^2. \quad (20)$$

Hence

$$\frac{d}{d\tau}(u_\alpha u^\alpha) = 0 \implies \frac{du_\alpha}{d\tau} u^\alpha + u_\alpha \frac{du^\alpha}{d\tau} = 2u_\alpha \frac{du^\alpha}{d\tau} = 0. \quad (20a)$$

The integrand of Eq. (19) can be written as

$$\sqrt{u_\alpha u^\alpha} d\tau = \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}} d\tau = \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta}.$$

Introducing a monotonic parameter $s = s(\tau)$, we obtain

$$A = -mc \int_{s_1}^{s_2} \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}} ds, \quad (21)$$

where $x^\alpha = x^\alpha(s)$ and s is a monotonically increasing function of τ . Thus,

$$\tilde{L}(s) = -mc \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}}, \quad A = \int \tilde{L} ds.$$

The Euler–Lagrange equation is

$$\frac{d}{ds} \left(\frac{\partial \tilde{L}}{\partial (dx^\alpha/ds)} \right) - \frac{\partial \tilde{L}}{\partial x^\alpha} = 0. \quad (24)$$

For the free particle, the variational calculation gives

$$mc \frac{d}{ds} \left[\frac{dx^\alpha/ds}{\sqrt{g_{\mu\nu} (dx^\mu/ds) (dx^\nu/ds)}} \right] = 0.$$

Using the normalization condition $u_\alpha u^\alpha = c^2$, we may choose $s = \tau$, so that

$$\sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}} = c,$$

and consequently

$$\frac{d^2 x^\alpha}{d\tau^2} = 0,$$

which is the equation of motion of a free particle.

For a charged particle in an external field, introduce the interaction term. The action becomes

$$A = - \int_{s_1}^{s_2} \left[mc \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}} + \frac{e}{c} \frac{dx^\alpha}{ds} A_\alpha(x) \right] ds. \quad (23)$$

Thus, the “quadrivector Lagrangian” is

$$\tilde{L} = - \left[mc \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}} + \frac{e}{c} \frac{dx^\alpha}{ds} A_\alpha(x) \right]. \quad (25)$$

The Euler–Lagrange equation gives

$$mc \frac{d}{ds} \left[\frac{dx^\alpha/ds}{mc^2 \sqrt{(dx^\mu/ds)(dx_\mu/ds)/c^2}} + \frac{e}{mc^2} A^\alpha \right] - \frac{e}{c} \frac{dx_\beta}{ds} \partial^\alpha A^\beta = 0,$$

or, choosing $s = \tau$,

$$m \frac{d^2 x^\alpha}{d\tau^2} + \frac{e}{c} \frac{dA^\alpha}{d\tau} - \frac{e}{c} \frac{dx_\beta}{d\tau} \partial^\alpha A^\beta = 0.$$

Since

$$\frac{dA^\alpha}{d\tau} = \frac{dx^\beta}{d\tau} \frac{\partial A^\alpha}{\partial x^\beta} = \frac{dx^\beta}{d\tau} \partial_\beta A^\alpha,$$

the equation of motion becomes

$$m \frac{d^2 x^\alpha}{d\tau^2} = \frac{e}{c} (\partial^\alpha A^\beta - \partial^\beta A^\alpha) \frac{dx_\beta}{d\tau}. \quad (26)$$

This is the equation of motion

$$m \frac{du^\alpha}{d\tau} = \frac{e}{c} F^{\alpha\beta} u_\beta, \quad (77)$$

i.e. the relativistic Lorentz force.

Hamiltonian formulation. The canonical momentum is

$$p^\alpha = -\frac{\partial \tilde{L}}{\partial(dx_\alpha/ds)} = mu^\alpha + \frac{e}{c} A^\alpha. \quad (27)$$

Hence,

$$u^\alpha = \frac{1}{m} \left(p^\alpha - \frac{e}{c} A^\alpha \right).$$

Defining the Hamiltonian as in the notes,

$$\tilde{H} = \frac{1}{2} \left(p_\alpha u^\alpha + \tilde{L} \right), \quad (28)$$

one obtains

$$\tilde{H} = \frac{1}{2m} \left(p_\alpha - \frac{e}{c} A_\alpha \right) \left(p^\alpha - \frac{e}{c} A^\alpha \right) - mc^2. \quad (29)$$

The Hamilton equations then give

$$\frac{dx^\alpha}{d\tau} = \frac{\partial \tilde{H}}{\partial p_\alpha} = \frac{1}{m} \left(p^\alpha - \frac{e}{c} A^\alpha \right), \quad \frac{dp^\alpha}{d\tau} = -\frac{\partial \tilde{H}}{\partial x_\alpha} = \frac{e}{mc} \left(p_\beta - \frac{e}{c} A_\beta \right) \partial^\alpha A^\beta. \quad (30)$$

The second equation is the relativistic Lorentz force equation.

3. Motion in a Uniform and Static Magnetic Field

For $\vec{B}$ uniform and static, the equations of motion are

$$\frac{d\vec{p}}{dt} = m\gamma \frac{d\vec{u}}{dt} = \frac{q}{c} \vec{u} \times \vec{B}, \quad \frac{dE}{dt} = \vec{F} \cdot \vec{u} = 0. \quad (37)$$

Thus, $E = \text{const.} \Rightarrow u = \text{const.} \Rightarrow r = \text{const.}$ Then

$$\frac{d\vec{u}}{dt} = \vec{u} \times \left(\frac{q\vec{B}}{\gamma mc} \right) \equiv \vec{u} \times \vec{\omega}_B, \quad (38)$$

where

$$\vec{\omega}_B = \frac{q\vec{B}}{\gamma mc} = \frac{qc\vec{B}}{\gamma mc^2} = \frac{qc\vec{B}}{E}, \quad (39)$$

and ω_B is the frequency of precession or gyration.

Let

$$\vec{B} = B\hat{e}_3, \quad \vec{u} = u_{\parallel}\hat{e}_3 + \vec{u}_{\perp}.$$

Then, from Eq. (38),

$$\frac{d\vec{u}}{dt} = (u_{\parallel}\hat{e}_3 + \vec{u}_{\perp}) \times \frac{qcB}{E}\hat{e}_3 = \vec{u}_{\perp} \times \vec{\omega}_B = \frac{d\vec{u}_{\perp}}{dt}.$$

Thus $u_{\parallel}$ is a constant of the motion. For the other components,

$$\frac{du_1}{dt} = \omega_B u_2, \quad \frac{du_2}{dt} = -\omega_B u_1,$$

which combine to give

$$\frac{d^2 u_1}{dt^2} = \frac{d}{dt}(\omega_B u_2) = -\omega_B^2 u_1.$$

A general solution is

$$u_1 = u_0 e^{-i\omega_B t},$$

with u_0 a constant. Hence

$$\frac{du_1}{dt} = -i\omega_B u_0 e^{-i\omega_B t} = \omega_B u_2 \implies u_2 = -iu_1.$$

Therefore,

$$\vec{u} = u_{\parallel}\hat{e}_3 + u_0(\hat{e}_1 - i\hat{e}_2)e^{-i\omega_B t}. \quad (40)$$

Since the velocity of rotation is uniform,

$$u_0 = \frac{2\pi a}{T} = \omega_B a,$$

where a is the radius of gyration. Integrating, we obtain

$$\vec{x}(t) = \vec{x}_0 + u_{\parallel}t\hat{e}_3 + ia(\hat{e}_1 - i\hat{e}_2)e^{-i\omega_B t}. \quad (41)$$

This is the equation of a helical trajectory of radius a and pitch angle

$$\alpha = \tan^{-1} \left(\frac{u_{\parallel}}{\omega_B a} \right).$$

Since

$$u_{\perp} = \omega_B a = \frac{qBa}{\gamma mc},$$

we have

$$\frac{u_{\perp}E}{c^2} = \frac{qBa}{c},$$

and, for the transverse momentum,

$$p_{\perp} = \frac{q}{c}Ba.$$

Using

$$1 \text{ kV} = 1.6 \times 10^{-19} \text{ C},$$

we obtain

$$1 \text{ kV}/c = 5.34 \times 10^{-28} \text{ kg m/s},$$

and, since

$$1 \text{ kg } \mu\text{s} = 1.87 \times 10^2 \text{ eV}/c = \frac{3 \times 10^8}{1.6 \times 10^{-19}} \text{ eV}/c,$$

we find

$$p_{\perp} = \frac{q}{c} Ba \Big|_{\mu\text{eV}} = \frac{qBa}{10^4 10^2 \text{ SI}} \frac{3 \times 10^8 \text{ eV}/c}{e} = 300 \left(\frac{q}{e} \right) Ba \text{ eV}/c.$$

Therefore, if $q = e$,

$$p_{\perp} (\text{MeV}/c) = 3 \times 10^{-4} B(\text{G}) a(\text{cm}). \quad (42)$$

5. Drift of Particles in a Static Non-Uniform Magnetic Field

Let $\vec{B}$ vary slowly:

$$\delta \vec{B} \quad \text{over} \quad \delta x \gg a \quad (\text{radius of gyration}).$$

First approximation: local treatment around $\vec{B}$.

Let $\hat{n}$ be such that

$$\vec{\nabla} B = \xi \hat{n}$$

at the point, with

$$\hat{n} \cdot \vec{B} = 0.$$

Then, within the first-order approximation,

$$\omega_B(\vec{x}) = \frac{e}{\gamma mc} B(\vec{x}) \simeq \omega_0 \left[1 + \frac{1}{B_0} \left(\frac{\partial B}{\partial \xi} \right)_0 \hat{n} \cdot \vec{x} \right]. \quad (47)$$

Here ξ is the coordinate in the direction $\hat{n}$.

The velocity is

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp},$$

with the motion locally helical. Let

$$\vec{v}_{\perp} = \vec{v}_0 + \vec{v}_1,$$

where $\vec{v}_0 \perp \vec{B}$ and $\vec{v}_1$ is a correction. From Eq. (47),

$$\frac{d\vec{v}_{\perp}}{dt} = \vec{v}_{\perp} \times \vec{\omega}_B(\vec{x}). \quad (48)$$

Therefore,

$$\frac{d\vec{v}_{\perp}}{dt} = \frac{d\vec{v}_0}{dt} + \frac{d\vec{v}_1}{dt} \simeq \vec{v}_{\perp} \times \vec{\omega}_0 + \vec{v}_0 (\hat{n} \cdot \vec{x}_0) \frac{1}{B_0} \left(\frac{\partial B}{\partial \xi} \right)_0 \times \vec{\omega}_0 + \dots,$$

and, to first order,

$$\frac{d\vec{v}_1}{dt} \simeq \left[\vec{v}_1 + \vec{v}_0 (\hat{n} \cdot \vec{x}_0) \frac{1}{B_0} \left(\frac{\partial B}{\partial \xi} \right)_0 \right] \times \vec{\omega}_0. \quad (49)$$

From Eq. (40),

$$\vec{v}(t) = v_{\parallel} \hat{e}_3 + \omega_B a (\hat{e}_1 - i \hat{e}_2) e^{-i \omega_B t},$$

so that

$$\vec{v}_0 = -\vec{\omega}_0 \times (\vec{x}_0 - \vec{X}).$$

From Eq. (41),

$$\vec{x}(t) = \vec{X} + v_{\parallel} t \hat{e}_3 + i a (\hat{e}_1 - i \hat{e}_2) e^{-i \omega_B t},$$

and therefore

$$\vec{x}_0 - \vec{X} = \frac{1}{\omega_0^2} (\vec{\omega}_0 \times \vec{v}_0). \quad (50)$$

Here $\vec{X}$ is the center of the gyration of the unperturbed circular motion, i.e.,

$$\vec{X} = 0$$

in this case. Thus,

$$\frac{d\vec{v}_1}{dt} \simeq \left[\vec{v}_1 - \frac{1}{B_0} \left(\frac{\partial B}{\partial \xi} \right)_0 \vec{\omega}_0 \times \vec{x}_0 (\hat{n} \cdot \vec{x}_0) \right] \times \vec{\omega}_0. \quad (51)$$

After eliminating the oscillatory term, $\vec{v}_1$ has a non-zero mean value:

$$\vec{v}_G = \langle \vec{v}_1 \rangle = \frac{1}{B_0} \left(\frac{\partial B}{\partial \xi} \right)_0 \vec{\omega}_0 \times \langle |(\vec{x}_0)_{\perp}| (\hat{n} \cdot \vec{x}_0) \rangle. \quad (52)$$

Assuming

$$\langle (\vec{x}_0)_{\perp} (\hat{n} \cdot \vec{x}_0) \rangle = \frac{a^2}{2} \hat{n}, \quad (53)$$

we obtain

$$\vec{v}_G = \frac{a^2}{2B_0} \left(\frac{\partial B}{\partial \xi} \right)_0 (\vec{\omega}_0 \times \hat{n}). \quad (54)$$

Equivalently,

$$\frac{\vec{v}_G}{\omega_B a} = \frac{a}{2B^2} (\vec{B} \times \vec{\nabla}_{\perp} B). \quad (55)$$

If

$$\left| \frac{\vec{\nabla} B}{\vec{B}} \right| \ll 1,$$

then

$$v_G \ll \omega_B a \quad (\text{the drift is much smaller than the orbital velocity}).$$

The drift direction is illustrated schematically in the notes for a positive charge and for a negative charge.

Now consider a curved magnetic field. Let the radius of curvature be

$$R \gg a.$$

The center of the spiral accompanies the curvature of $\vec{B}$; therefore, it has a centripetal acceleration

$$\frac{v_{\parallel}^2}{R},$$

which is equivalent to an effective electric force:

$$eE_{\text{eff}} = \gamma m \frac{v_{\parallel}^2}{R}. \quad (56)$$

In addition to B_0 , Eq. (43) gives the drift velocity

$$\vec{v}_C \simeq c \frac{\gamma m}{e} \frac{v_{\parallel}^2}{R} \frac{\vec{R} \times \vec{B}_0}{R^2 B_0^2}. \quad (57)$$

Or,

$$\vec{v}_C = \frac{v_{\parallel}^2}{\omega_B R} \left(\frac{\vec{R} \times \vec{B}_0}{R B_0} \right). \quad (58)$$

The latter result can be obtained directly by solving the equation of motion for the Lorentz force in cylindrical coordinates,

$$\vec{B} = B_{\phi} \hat{\phi} = B_0 \hat{\phi},$$

which gives

$$\ddot{\rho} - \rho \dot{\phi}^2 = -\omega_B \dot{z}, \quad \rho \ddot{\phi} + 2\dot{\rho} \dot{\phi} = 0, \quad \ddot{z} = \omega_B \dot{\rho}. \quad (59)$$

For $R \gg a$, keeping only the lowest-order terms,

$$\dot{\phi} \simeq \frac{v_{\parallel}}{R}, \quad \rho \simeq R.$$

The first equation in (59) then gives

$$\ddot{z} \simeq \frac{v_{\parallel}^2}{\omega_B R}, \quad (60)$$

which is the same as $\vec{v}_C$ in Eq. (58).

In a current-free region, we can combine Eqs. (55) and (58):

$$\vec{\nabla} \times \vec{B} = 0 \quad \implies \quad \frac{\vec{\nabla}_{\perp} B}{B} = -\frac{\vec{R}}{R^2}.$$

Substitution into Eq. (55) gives

$$\vec{v}_G = \frac{\omega_B a^2}{2} \left[\frac{\vec{B}}{B} \times \left(-\frac{\vec{R}}{R^2} \right) \right] = \frac{v_{\perp}^2}{2\omega_B R} \left(\frac{\vec{R} \times \vec{B}}{R B} \right).$$

The drifts $\vec{v}_G$ and $\vec{v}_C$ are parallel and therefore can be added directly. The total drift velocity is

$$\vec{v}_D = \frac{1}{\omega_B R} \left(v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right) \left(\frac{\vec{R} \times \vec{B}}{R B} \right). \quad (61)$$

8. Lagrangian for the Electromagnetic Field

Lagrangian for the electromagnetic field in interaction with charge and current. A field in general has infinitely many degrees of freedom.

$$i \longrightarrow x^\alpha \quad k \longrightarrow \phi_k(x)$$

Generalized coordinates:

$$q_i \longrightarrow \phi_k(x),$$

where each point x^α corresponds to a set of variables x_i .

Generalized velocities:

$$\dot{q}_i \longrightarrow \partial^\alpha \phi_k(x).$$

Thus,

$$L = \sum_i L_i(q_i, \dot{q}_i) \longrightarrow \int \mathcal{L}(\phi_k, \partial^\alpha \phi_k) d^3x. \quad (83)$$

The Euler–Lagrange equation becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \quad \longrightarrow \quad \partial^\beta \left(\frac{\partial \mathcal{L}}{\partial (\partial^\beta \phi_k)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_k}.$$

Here $\mathcal{L}$ is the Lagrangian density.

The action integral is

$$A = \int \mathcal{L} d^3x dt = \int \mathcal{L} d^4x. \quad (84)$$

If A and d^4x are Lorentz invariants, then $\mathcal{L}$ is invariant.

Requiring that the action be an extremum with respect to variations of the field,

$$\frac{\delta A}{\delta \phi_k(x)} = 0,$$

we obtain the Euler–Lagrange equation

$$\partial^\beta \left(\frac{\partial \mathcal{L}}{\partial (\partial^\beta \phi_k)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_k} = 0.$$

From the quantities $F^{\alpha\beta}$, A^α , and J^α we construct the scalar

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha. \quad (85)$$

The interaction term is consistent with Eq. (10):

$$\mathcal{L}_{\text{int}} = -e\Phi + \frac{e}{c} \vec{u} \cdot \vec{A}.$$

Using Eqs. (11) and (136),

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha,$$

and therefore

$$\mathcal{L} = -\frac{1}{16\pi} (\partial^\alpha A^\beta - \partial^\beta A^\alpha) (\partial_\alpha A_\beta - \partial_\beta A_\alpha) - \frac{1}{c} J_\alpha A^\alpha. \quad (86)$$

Calculating,

$$\frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)} = -\frac{1}{4\pi} F_{\alpha\beta} = \frac{1}{4\pi} F_{\beta\alpha}. \quad (87)$$

Also,

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c} J_\alpha. \quad (88)$$

The equations of motion are therefore

$$\frac{1}{4\pi} \partial^\beta F_{\beta\alpha} = \frac{1}{c} J_\alpha, \quad (89)$$

or

$$\partial_\beta F^{\beta\alpha} = \frac{4\pi}{c} J^\alpha,$$

which are the covariant form of Maxwell's equations.

The homogeneous equations are automatically satisfied:

$$\partial_\alpha {}^\star F^{\alpha\beta} = \frac{1}{2} \partial_\alpha \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} = \frac{1}{2} \partial_\alpha \epsilon^{\alpha\beta\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) = 0,$$

because the Levi-Civita tensor is antisymmetric while the second derivatives are symmetric.

The continuity equation follows from the divergence of Eq. (89):

$$\frac{1}{4\pi} \partial_\alpha \partial_\beta F^{\beta\alpha} = \frac{1}{c} \partial_\alpha J^\alpha \implies \partial_\alpha J^\alpha = 0. \quad (90)$$

9. Proca Lagrangian: Effect of the Photon Mass

Add a mass term to Eq. (85):

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha - \frac{1}{c} J_\alpha A^\alpha, \quad (91)$$

where μ is the inverse of the Compton wavelength of the photon:

$$\mu = \frac{m_\gamma c}{\hbar}.$$

The equation of motion is

$$\partial_\beta F^{\beta\alpha} + \mu^2 A^\alpha = \frac{4\pi}{c} J^\alpha. \quad (92)$$

Using

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A},$$

and

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha,$$

where

$$\partial^\alpha = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right), \quad A^{\alpha,\beta} = (\Phi, \vec{A}),$$

the Lorenz condition is

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0 \quad \implies \quad \partial_\alpha A^\alpha = 0.$$

Calculating the first term of Eq. (92),

$$\begin{aligned} \partial_\beta F^{\beta\alpha} &= \partial_\beta (\partial^\beta A^\alpha - \partial^\alpha A^\beta) \\ &= \partial_\beta \partial^\beta A^\alpha - \partial^\alpha \partial_\beta A^\beta \\ &= \square A^\alpha, \end{aligned}$$

and hence

$$\square A_\alpha + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha. \quad (93)$$

In the static limit,

$$\frac{\partial \vec{A}}{\partial t} = 0,$$

and therefore

$$\nabla^2 A_\alpha - \mu^2 A_\alpha = -\frac{4\pi}{c} J_\alpha.$$

For a point charge q at the origin, only the temporal component of the potential, $A_0 = \Phi$, is nonzero, and it assumes the Yukawa form:

$$\Phi(x) = \frac{q}{r} e^{-\mu r}. \quad (94)$$

Now, if the photon has mass,

$$E^2 = (\hbar\omega)^2 = p^2 c^2 + m_\gamma^2 c^4 = \frac{\hbar^2}{\lambda^2} (2\pi)^2 c^2 + m_\gamma^2 c^4,$$

so that

$$\omega^2 = \left(\frac{2\pi c}{\lambda} \right)^2 + \mu^2 c^2 \equiv \omega_0^2 + \mu^2 c^2. \quad (95,96)$$

10. Canonical and Symmetric Stress Tensors; Conservation Laws

a) Generalization of the Hamiltonian: canonical stress tensor

Defining the canonical variables of the momentum

$$p^i = \frac{\partial L}{\partial \dot{q}_i}, \quad (99)$$

and the Hamiltonian

$$H = \sum_i p_i \dot{q}_i - L, \quad (100)$$

if $\partial L / \partial t = 0$, then $dH / dt = 0$, and H is a constant of the motion.

Now, the energy is the temporal component of the energy-momentum four-vector; we want the Hamiltonian to be transformed in an identical manner, as:

$$H = \int \mathcal{H} d^3x = \int \mathcal{H} \frac{d^3x}{dx^0} dx^0. \quad (78)$$

Since

$$d^4x = d^3x dx^0,$$

$\mathcal{H}$ transforms as the component $(0,0)$ of a second-rank tensor.

Moreover, since

$$\mathcal{L} = \mathcal{L}[\phi_k(x), \partial^\alpha \phi_k(x)]$$

for the fields, we introduce, in analogy with Eq. (100),

$$\mathcal{H} = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_k)} \frac{\partial \phi_k}{\partial t} - \mathcal{L}. \quad (101)$$

Extending the time derivatives to $\partial_\alpha \phi_k$ gives the generalization of the Hamiltonian density, with the canonical stress tensor:

$$T^{\alpha\beta} = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k - g^{\alpha\beta} \mathcal{L}. \quad (102)$$

In the absence of sources ($J^\alpha = 0$), the Lagrangian density of the free electromagnetic field is

$$\mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}. \quad (103)$$

Substituting A^λ for ϕ_k (with repeated indices understood),

$$T^{\alpha\beta} = 2 \frac{\partial \mathcal{L}_{\text{em}}}{\partial(\partial_\alpha A_\lambda)} \partial^\beta A_\lambda - g^{\alpha\beta} \mathcal{L}_{\text{em}}.$$

Since

$$\mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{16\pi} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu),$$

we obtain

$$\frac{\partial \mathcal{L}_{\text{em}}}{\partial(\partial_\alpha A^\lambda)} = -\frac{1}{16\pi} (\delta_\mu^\alpha \delta_\nu^\lambda - \delta_\nu^\alpha \delta_\mu^\lambda) F^{\mu\nu} - \frac{1}{16\pi} F_{\mu\nu} (g^{\mu\alpha} g^{\nu\lambda} - g^{\nu\alpha} g^{\mu\lambda}),$$

and hence

$$\frac{\partial \mathcal{L}_{\text{em}}}{\partial(\partial_\alpha A^\lambda)} = -\frac{1}{16\pi} (F^{\alpha\lambda} + F^{\alpha\lambda} + g^{\alpha\mu} F_\mu{}^\lambda + g^{\alpha\nu} F_\nu{}^\lambda) = -\frac{1}{4\pi} F^\alpha{}_\lambda.$$

Thus

$$T^{\alpha\beta} = -\frac{1}{4\pi} F^\alpha{}_\lambda \partial^\beta A^\lambda - g^{\alpha\beta} \mathcal{L}_{\text{em}}. \quad (104)$$

Using the electromagnetic tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix},$$

we have

$$\mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = \frac{1}{8\pi} (E^2 - B^2). \quad (79)$$

For the particular component T^{00} ,

$$T^{00} = -\frac{1}{4\pi} F_{0\lambda} \partial^0 A^\lambda - \frac{1}{8\pi} (E^2 - B^2) = -\frac{1}{4\pi} \left(\vec{E} \cdot \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) - \frac{1}{8\pi} (E^2 - B^2).$$

Since

$$\vec{E} = -\vec{\nabla} \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t},$$

we have

$$-\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = \vec{E} + \vec{\nabla} \Phi,$$

and therefore

$$T^{00} = \frac{E^2 + B^2}{8\pi} + \frac{1}{4\pi} \vec{E} \cdot \vec{\nabla} \Phi. \quad (105)$$

In a source-free region,

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \implies \quad \vec{\nabla} \cdot (\vec{E} \Phi) = \vec{E} \cdot \vec{\nabla} \Phi,$$

and in a localized region,

$$\int_V \vec{\nabla} \cdot (\vec{E} \Phi) d^3x = \oint_S (\vec{E} \Phi) \cdot \hat{n} da = 0.$$

Thus

$$\int_V T^{00} d^3x = \frac{1}{8\pi} \int_V (E^2 + B^2) d^3x \equiv \mathcal{E}_{\text{field}}. \quad (106)$$

What about the other components of the stress tensor? Analogously,

$$T^{0i} = \frac{1}{4\pi} \left[(\vec{E} \times \vec{B})_i + \vec{\nabla} \cdot (A_i \vec{E}) \right], \quad (105')$$

and

$$T^{i0} = \frac{1}{4\pi} \left[(\vec{E} \times \vec{B})_i + \frac{1}{c} \left((\vec{\nabla} \times \vec{B})_i \Phi - 2 \frac{\partial (\vec{E} \cdot \vec{A})}{\partial x^0} \right) \right]. \quad (105'')$$

The tensor is not a symmetric tensor; its divergence in Eq. (105'') implies that also for T^{0i} ,

$$\int_V T^{0i} d^3x = \frac{1}{4\pi} \int_V (\vec{E} \times \vec{B})_i d^3x \equiv P_{\text{field}}^i. \quad (106')$$

Equations (106') represent a covariant generalization of Poynting's theorem:

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0. \quad (80)$$

We will now show that

$$\partial_\alpha T^{\alpha\beta} = 0. \quad (107)$$

Starting from Eq. (102),

$$\begin{aligned}\partial_\alpha T^{\alpha\beta} &= \sum_k \partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k \right] - \partial^\beta \mathcal{L} \\ &= \sum_k \left[\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \right) \partial^\beta \phi_k + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\alpha \partial^\beta \phi_k \right] - \partial^\beta \mathcal{L}.\end{aligned}$$

Using the Euler–Lagrange equation,

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_k},$$

we get

$$\partial_\alpha T^{\alpha\beta} = \sum_k \left[\frac{\partial \mathcal{L}}{\partial \phi_k} \partial^\beta \phi_k + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\alpha \partial^\beta \phi_k \right] - \partial^\beta \mathcal{L} = 0.$$

Therefore,

$$\frac{d\mathcal{E}_{\text{field}}}{dt} = 0, \quad \frac{d\vec{P}_{\text{field}}}{dt} = 0. \quad (108)$$

The densities of energy and momentum are

$$u = \frac{1}{8\pi}(E^2 + B^2), \quad \vec{S} = \frac{1}{4\pi c}(\vec{E} \times \vec{B}).$$

b) The symmetric form of the stress tensor

The canonical stress tensor obtained above, $T^{\alpha\beta}$, is convenient, but has the disadvantage of not being symmetric (see Eq. (105').) The question of symmetry arises when we analyze the angular momentum of the field:

$$\vec{L}_{\text{field}} = \frac{1}{4\pi c} \int \vec{x} \times (\vec{E} \times \vec{B}) d^3x.$$

The covariant generalization of the angular-momentum density is a third-rank tensor:

$$M^{\alpha\beta\gamma} = T^{\alpha\beta} x^\gamma - T^{\alpha\gamma} x^\beta. \quad (109)$$

Its conservation law states that the four-divergence vanishes:

$$\partial_\alpha M^{\alpha\beta\gamma} = 0. \quad (110)$$

But

$$\begin{aligned}\partial_\alpha M^{\alpha\beta\gamma} &= \partial_\alpha (T^{\alpha\beta} x^\gamma - T^{\alpha\gamma} x^\beta) \\ &= (\partial_\alpha T^{\alpha\beta}) x^\gamma + T^{\alpha\beta} \partial_\alpha x^\gamma - (\partial_\alpha T^{\alpha\gamma}) x^\beta - T^{\alpha\gamma} \partial_\alpha x^\beta \\ &= T^{\beta\gamma} - T^{\gamma\beta} \neq 0,\end{aligned}$$

for the canonical stress tensor.

Furthermore, $T^{\alpha\beta}$ depends on the potentials and is not invariant under gauge transformations, and its trace is not zero.

c) Conservation law in the presence of sources

Consider

$$\partial_\alpha T^{\alpha\beta} = \frac{1}{4\pi} (\partial_\alpha F^{\alpha\mu}) F^\beta{}_\mu - \frac{1}{4\pi} F^{\alpha\mu} \partial_\alpha \partial^\beta A_\mu. \quad (81)$$

Using Maxwell's equation,

$$\frac{1}{4\pi} \partial_\alpha F^{\alpha\mu} = \frac{1}{c} J^\mu,$$

we obtain

$$\partial_\alpha T^{\alpha\beta} + F^{\beta\mu} \frac{J_\mu}{c} = \frac{1}{8\pi} F_{\mu\nu} (\partial^\mu F^{\beta\nu} + \partial^\nu F^{\mu\beta} + \partial^\beta F^{\nu\mu}) = 0,$$

where the last equality follows from the antisymmetry of $F_{\mu\nu}$ and the homogeneous Maxwell equation.

Thus,

$$\partial_\alpha T^{\alpha\beta} = -\frac{1}{c} F^{\beta\mu} J_\mu. \quad (118)$$

The temporal component is

$$\frac{1}{c} \left(\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} \right) = -\frac{1}{c} \vec{J} \cdot \vec{E}, \quad (119)$$

which is Poynting's theorem in the presence of sources.

The spatial component is

$$\frac{\partial S_i/c}{\partial t} - \sum_{j=1}^3 \frac{\partial T_{ij}^{(M)}}{\partial x_j} = - \left[\rho E_i + \frac{1}{c} (\vec{J} \times \vec{B})_i \right], \quad (120)$$

where

$$J^\alpha = (c\rho, \vec{J}).$$

The four-vector in the second member of Eq. (108) is the density of Lorentz force:

$$f^\beta \equiv \frac{1}{c} F^{\beta\alpha} J_\alpha = \left(\frac{1}{c} \vec{J} \cdot \vec{E}, \rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} \right). \quad (121)$$

Its integral gives the equation of motion of the particles:

$$\int f^i d^3x = \frac{d}{dt} \vec{P}_{\text{particles}}.$$

Thus,

$$\int d^3x (\partial_\alpha T^{\alpha\beta} + f^\beta) = \frac{d}{dt} (P_{\text{field}}^\beta + P_{\text{particles}}^\beta) = 0, \quad (82)$$

which is the conservation of four-momentum for the system (field + particles).

Chapter 13 – Charged-Particle Collisions, Energy Loss, and Scattering

1. Energy Transfer in a Coulomb Collision

Consider a fast charged particle passing near another charged particle. Heavy particles ($\mu, \pi, K, p, \dots$) are approximately described by rectilinear trajectories in the present treatment, whereas electrons require special attention because of their comparatively large deflections.

1. Energy transfer in a Coulomb collision

Consider a projectile of charge $q = 3e$, mass $M \gg m$, and velocity v , incident on an electron initially at rest. Let b be the impact parameter. We assume that the incident particle is fast, $v \gg v_e$, so that the electron may be regarded as initially free during the collision. The momentum transfer to the electron is predominantly transverse.

The electric field at the position of the electron due to the incident particle is

$$E_1 = -\frac{qvt}{(b^2 + y^2 + v^2t^2)^{3/2}}, \quad E_2 = \frac{qb}{(b^2 + y^2 + v^2t^2)^{3/2}}. \quad (1)$$

For the collision geometry considered here, the transverse impulse is

$$\Delta p = e \int_{-\infty}^{+\infty} E_2(t) dt = \frac{e}{v} \int_{-\infty}^{+\infty} E_2(x) dx. \quad (83)$$

Using Gauss's law for a cylindrical surface, the flux through the cylinder is $2\pi b dx$, and therefore

$$\int_{-\infty}^{+\infty} E_2(x) dx = \frac{2q}{b}. \quad (84)$$

Consequently,

$$\Delta p = \int_{-\infty}^{+\infty} eE_2(t) dt = \frac{2qe}{bv} = \frac{2(3e)e}{bv} = \frac{6e^2}{bv}. \quad (85)$$

The energy transferred to the electron is, in the nonrelativistic approximation for the electron,

$$\Delta E(b) = \frac{\Delta p^2}{2m} = \frac{2 \cdot 3^2 e^4}{mv^2} \frac{1}{b^2}. \quad (2)$$

For a small scattering angle, $\theta \ll 1$,

$$\tan \theta \simeq \theta \simeq \frac{\Delta p}{p} = \frac{2Ze^2}{mv^2b}. \quad (3)$$

More generally, the Rutherford trajectory in polar coordinates satisfies

$$\frac{1}{r} = \frac{1}{b} \sin \varphi + \frac{d}{2b^2} (\cos \varphi - 1), \quad (86)$$

where

$$d = \frac{3e^2}{mv^2/2} \quad (87)$$

is the distance of closest approach for a head-on elastic collision.

For large distances,

$$\sin \varphi = \sin \theta, \quad \cos \varphi = -\cos \theta, \quad \varphi + \theta = \pi, \quad (88)$$

and, as $r \rightarrow \infty$,

$$0 = \frac{1}{b} \sin \theta - \frac{d}{2b^2} (\cos \theta + 1). \quad (89)$$

Thus,

$$\frac{\cos \theta + 1}{\sin \theta} = \cot \frac{\theta}{2} = \frac{2b}{d} = \frac{2b}{3e^2/(mv^2/2)}, \quad (90)$$

or

$$\tan \frac{\theta}{2} = \frac{3e^2}{mv^2b}. \quad (4)$$

For $\theta \ll 1$, $p = mv$ and $Z = 1$, this is consistent with Eq. (3).

Now consider a head-on collision between the projectile of mass M and the electron.

Relativistic energy conservation gives

$$\sqrt{p_i^2 c^2 + M^2 c^4} + mc^2 = \sqrt{p_f^2 c^2 + M^2 c^4} + K + mc^2, \quad (91)$$

where $K = E_e - mc^2$ is the kinetic energy acquired by the electron. Momentum conservation gives

$$p_i = p_f + p_e, \quad (92)$$

and therefore

$$p_f^2 = p_i^2 + p_e^2 - 2p_i p_e \cos \theta. \quad (93)$$

The maximum energy transfer occurs for a head-on collision, $\theta = 0$, leading to

$$K_{\max} = \frac{2mc^2 p_i^2 c^2}{\left(mc^2 + \sqrt{p_i^2 c^2 + M^2 c^4} \right)^2 - p_i^2 c^2}. \quad (94)$$

For $M \gg m$,

$$K_{\max} = 2mc^2(\gamma^2 - 1) = 2m\gamma^2v^2. \quad (5)$$

Equating Eq. (2) to the maximum transferable energy gives

$$\frac{2\mathcal{Z}^2e^4}{mv^2b_{\min}^2} = 2m\gamma^2v^2, \quad (95)$$

and hence

$$b_{\min} = \frac{3e^2}{\gamma mv^2}. \quad (6)$$

Problem 13.1

For a collision with an electron initially at rest, the electron momentum components after the collision are

$$p'_{\perp} = mv \sin \theta, \quad p'_{\parallel} = mv(\cos \theta - 1). \quad (96)$$

Therefore,

$$p^2 = m^2v^2(1 - \cos \theta), \quad (97)$$

and the transferred energy is

$$\Delta E = \frac{p^2}{2m} = mv^2(1 - \cos \theta) = 2mv^2 \sin^2 \frac{\theta}{2} = 2mv^2 \frac{\tan^2(\theta/2)}{1 + \tan^2(\theta/2)}. \quad (98)$$

Using Eq. (4),

$$\tan \frac{\theta}{2} = \frac{3e^2}{mv^2b}, \quad (99)$$

so that

$$\sin^2 \frac{\theta}{2} = \left(\frac{3e^2}{mv^2} \right)^2 \frac{1}{(3e^2/(mv^2))^2 + b^2}. \quad (100)$$

Using Eq. (6), in the nonrelativistic limit,

$$b_{\min} \simeq \frac{3e^2}{mv^2}, \quad (101)$$

and consequently

$$\Delta E(b) = \frac{2\mathcal{Z}^2e^4}{mv^2} \left(\frac{1}{b_{\min}^2 + b^2} \right). \quad (7)$$

For $b \gg b_{\min}$, Eq. (7) reduces to Eq. (2).

For sufficiently large b , the binding of the electron to the atom must be taken into account; the electron is no longer effectively free. If the collision duration

$$\Delta t \simeq \frac{b_{\max}}{v} \quad (102)$$

is comparable to the period of oscillation of the electron in its orbit,

$$\frac{b_{\max}}{v} \simeq \frac{2\pi}{\omega} \equiv \tau, \quad (103)$$

in the reference frame of the incident particle of charge $3e$. Thus,

$$b_{\max} \simeq \frac{2\pi\gamma v}{\omega}. \quad (9)$$

The energy-loss curve therefore has a lower cutoff $b_{\min}$, associated with the maximum transferable energy, and an upper cutoff $b_{\max}$, associated with the finite binding/oscillation time of the atomic electron.

Now suppose that the incident particle traverses a medium containing electrons with number density NZ , where Z is the atomic number. The number of electrons with impact parameter between b and $b + db$ encountered over a distance dx is

$$dm = NZ 2\pi b db dx. \quad (10)$$

Hence,

$$\frac{dE}{dx} = 2\pi NZ \int \Delta E(b) b db. \quad (11)$$

Using Eq. (2) and integrating between $b_{\min}$ and $b_{\max}$,

$$\begin{aligned} \frac{dE}{dx} &\simeq 4\pi NZ \frac{3^2 e^4}{mv^2} \int_{b_{\min}}^{b_{\max}} \frac{db}{b} \\ &\simeq 4\pi NZ \frac{3^2 e^4}{mv^2} \ln B, \end{aligned} \quad (12)$$

where

$$B = \frac{b_{\max}}{b_{\min}} \simeq \frac{2\pi\gamma v}{\omega} \frac{mv^2}{3e^2} = \frac{2\pi\gamma mv^3}{3e^2\omega}. \quad (13-14)$$

This is the classical result of Bohr (1915).

2. Energy Transfer to a Harmonically Bound Charge

For Coulomb collisions,

$$\begin{cases} b < b_{\max} : & \text{energy transfer is given by Eq. (2),} \\ b > b_{\max} : & \text{adiabatic.} \end{cases} \quad (104)$$

Consider a heavy incident particle, $M \gg m$, with charge $3e$ and velocity $\mathbf{v}$, passing at an impact parameter b from an electron bound harmonically with natural frequency ω_0 . We assume

$$x_0 \ll b. \quad (105)$$

The electron is nonrelativistic.

The equation of motion for the harmonically bound charge is

$$\ddot{\mathbf{x}} + \Gamma \dot{\mathbf{x}} + \omega_0^2 \mathbf{x} = -\frac{e}{m} \mathbf{E}(t). \quad (15)$$

We use the Fourier convention

$$\mathbf{x}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathbf{x}(\omega) e^{-i\omega t} d\omega, \quad (16)$$

$$\mathbf{E}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathbf{E}(\omega) e^{-i\omega t} d\omega. \quad (17)$$

The inverse transforms are

$$\mathbf{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathbf{x}(t) e^{i\omega t} dt, \quad \mathbf{E}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathbf{E}(t) e^{i\omega t} dt. \quad (106)$$

Since $\mathbf{x}(t)$ and $\mathbf{E}(t)$ are real,

$$\mathbf{x}(-\omega) = \mathbf{x}^*(\omega), \quad \mathbf{E}(-\omega) = \mathbf{E}^*(\omega). \quad (18)$$

Substitution into Eq. (15), using

$$\dot{\mathbf{x}}(t) \longrightarrow -i\omega \mathbf{x}(\omega), \quad \ddot{\mathbf{x}}(t) \longrightarrow -\omega^2 \mathbf{x}(\omega), \quad (107)$$

gives

$$-\omega^2 \mathbf{x}(\omega) - i\omega \Gamma \mathbf{x}(\omega) + \omega_0^2 \mathbf{x}(\omega) = -\frac{e}{m} \mathbf{E}(\omega), \quad (108)$$

and hence

$$\mathbf{x}(\omega) = -\frac{e}{m} \frac{\mathbf{E}(\omega)}{\omega_0^2 - i\omega \Gamma - \omega^2}. \quad (19)$$

The work done by the incident particle on the bound charge is

$$\mathcal{P} = \frac{dE}{dt} = \mathbf{F} \cdot \frac{d\mathbf{x}}{dt} = \mathbf{F} \cdot \mathbf{v}. \quad (109)$$

Using

$$\begin{aligned} \mathcal{P} &= \int d^3x' \mathbf{F} \cdot \mathbf{v} = \int d^3x' \rho \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \mathbf{v} \\ &= \int d^3x' \mathbf{E} \cdot \rho \mathbf{v} = \int d^3x' \mathbf{E} \cdot \mathbf{J}, \end{aligned} \quad (110)$$

we obtain

$$\Delta E = \int_{-\infty}^{+\infty} dt \int d^3x' \mathbf{E} \cdot \mathbf{J}. \quad (21)$$

The current density of the electron is

$$\mathbf{J} = -e\dot{\mathbf{x}}\delta[\mathbf{x}' - \mathbf{x}(t)]. \quad (111)$$

Therefore,

$$\Delta E = -e \int_{-\infty}^{+\infty} dt \mathbf{E} \cdot \mathbf{v}, \quad \mathbf{v} = \dot{\mathbf{x}}. \quad (122)$$

Inserting the Fourier representations of $\mathbf{x}(t)$ and $\mathbf{E}(t)$, and using the reality relations (18), one obtains

$$\begin{aligned} \Delta E &= e \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' i\omega \mathbf{x}(\omega) \cdot \mathbf{E}^*(\omega') \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(\omega' - \omega)t} dt \right] \\ &= e \int_{-\infty}^{+\infty} d\omega i\omega \mathbf{x}(\omega) \cdot \mathbf{E}^*(\omega). \end{aligned} \quad (112)$$

Separating the positive- and negative-frequency parts,

$$\begin{aligned} \int_{-\infty}^0 d\omega i\omega \mathbf{x}(\omega) \cdot \mathbf{E}^*(\omega) &= \int_0^{+\infty} d\omega i\omega \mathbf{x}^*(\omega) \cdot \mathbf{E}(\omega) \\ &= \int_0^{+\infty} d\omega [i\omega \mathbf{x}(\omega) \cdot \mathbf{E}^*(\omega)]^*. \end{aligned} \quad (113)$$

Since $z^* + z = 2 \operatorname{Re} z$, it follows that

$$\Delta E = 2e \operatorname{Re} \int_0^{+\infty} i\omega \mathbf{x}(\omega) \cdot \mathbf{E}^*(\omega) d\omega. \quad (123)$$

Substituting Eq. (19),

$$\begin{aligned} \Delta E &= -2 \frac{e^2}{m} \operatorname{Re} \int_0^{+\infty} i\omega \frac{|\mathbf{E}(\omega)|^2}{\omega_0^2 - \omega^2 - i\omega\Gamma} d\omega \\ &= -2 \frac{e^2}{m} \operatorname{Re} \int_0^{+\infty} |\mathbf{E}(\omega)|^2 \frac{i\omega(\omega_0^2 - \omega^2 - i\omega\Gamma)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2} d\omega \\ &= \frac{e^2}{m} \int_0^{+\infty} |\mathbf{E}(\omega)|^2 \frac{2\omega^2\Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2} d\omega. \end{aligned} \quad (124)$$

The energy transfer is therefore concentrated around the natural frequency $\omega \simeq \omega_0$, with a Lorentzian line shape.

Approximating the electric field by its value at the natural frequency, $\mathbf{E}(\omega) \simeq \mathbf{E}(\omega_0)$, gives

$$\begin{aligned} \Delta E &\simeq \frac{2e^2}{m} |\mathbf{E}(\omega_0)|^2 \int_0^{+\infty} \frac{\omega^2\Gamma d\omega}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2} \\ &= \frac{2e^2}{m} |\mathbf{E}(\omega_0)|^2 \int_0^{+\infty} \frac{x^2 dx}{[(\omega_0/\Gamma)^2 - x^2]^2 x^2 + 1}, \end{aligned} \quad (125)$$

where

$$x = \frac{\omega}{\Gamma}. \quad (114)$$

For a narrow Lorentzian,

$$L(x) = \frac{1}{\pi} \frac{\Gamma/2}{(x - x_0)^2 + (\Gamma/2)^2}, \quad \int_{-\infty}^{+\infty} L(x) dx = 1, \quad (115)$$

and therefore

$$\int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{(\Gamma/2)}{[(y - y_0)^2 + (\Gamma/2)^2]} dy = 1. \quad (116)$$

Thus the energy transfer becomes

$$\Delta E = \frac{\pi e^2}{m} |\mathbf{E}(\omega_0)|^2. \quad (26)$$

We now calculate $|\mathbf{E}(\omega_0)|^2$.

For the geometry shown in the notes, the transverse component of the electric field is

$$E_2(\omega) = \frac{3eb\gamma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{i\omega t} dt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}. \quad (27)$$

With

$$x = \frac{\gamma v t}{b}, \quad (117)$$

this becomes

$$E_2(\omega) = \frac{3e}{\sqrt{2\pi} b v} \int_{-\infty}^{+\infty} \frac{e^{i\omega b x / (\gamma v)}}{(1 + x^2)^{3/2}} dx. \quad (28)$$

The integral is a modified Bessel function of the second kind of order one:

$$E_2(\omega) = \frac{3e}{bv} \sqrt{\frac{2}{\pi}} \left[\frac{\omega b}{\gamma v} K_1 \left(\frac{\omega b}{\gamma v} \right) \right]. \quad (29)$$

Similarly,

$$E_1(\omega) = -i \frac{3e}{\gamma v} \sqrt{\frac{2}{\pi}} \left[\frac{\omega b}{\gamma v} K_0 \left(\frac{\omega b}{\gamma v} \right) \right]. \quad (30)$$

Consequently, the energy transfer to the harmonically bound charge is

$$\Delta E(b) = \frac{2 \cdot 3^2 e^4}{m v^2} \frac{1}{b^2} \left[\xi^2 K_1^2(\xi) + \frac{1}{\gamma^2} \xi^2 K_0^2(\xi) \right], \quad (31)$$

where

$$\xi = \frac{\omega_0 b}{\gamma v} = \frac{b}{b_{\max}}. \quad (32)$$

The factor in square brackets has the limiting behaviors

$$[\dots] = \begin{cases} 1, & \xi \ll 1 \quad \text{or} \quad b \ll b_{\max}, \\ \left(1 + \frac{1}{\gamma^2}\right) \frac{\pi}{2} \xi e^{-2\xi}, & \xi \gg 1 \quad \text{or} \quad b \gg b_{\max}. \end{cases} \quad (33)$$

Thus the energy transfer approaches the result for a free electron for $b \ll b_{\max}$, whereas it is exponentially suppressed for $b \gg b_{\max}$, corresponding to the adiabatic regime.

3. Classical and Quantum Formulas for Energy Loss

Suppose N atoms per unit volume, with Z electrons per atom, and let the j th electron have frequency ω_j . Then

$$\sum_j f_j = Z, \quad (118)$$

where f_j is the oscillator-strength constant of the j th oscillator. Thus,

$$\frac{dE}{dx} = 2\pi N Z \sum_j f_j \int_{b_{\min}}^{\infty} \Delta E_j(b) b db \quad (34)$$

$$\begin{aligned} &= \frac{4\pi N Z 3^2 e^4}{mv^2} \sum_j f_j \int_{b_{\min}}^{\infty} \left[\frac{\xi^2 K_1^2(\xi)}{b^2} + \frac{\xi^2 K_0^2(\xi)}{\gamma^2 b^2} \right] b db \\ &= \frac{4\pi N Z 3^2 e^4}{mv^2} \sum_j f_j \int_{\xi_{\min}}^{\infty} \left[K_1^2(\xi) + \frac{\xi^2}{\gamma^2} K_0^2(\xi) \right] \frac{d\xi}{\xi}. \end{aligned} \quad (119)$$

Remembering the formulas for the derivatives of the modified Bessel functions,

$$K'_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x} K_\nu(x), \quad K'_\nu(x) = -K_{\nu+1}(x) + \frac{\nu}{x} K_\nu(x), \quad (120)$$

we have

$$\begin{aligned} \frac{d}{dx} [x K_0 K_1] &= K_0 K_1 + x (K'_0 K_1 + K_0 K'_1) \\ &= K_0 K_1 + x \left[-K_1^2 + K_0 \left(-K_0 - \frac{1}{x} K_1 \right) \right] \\ &= -x (K_0^2 + K_1^2), \end{aligned} \quad (121)$$

and

$$\begin{aligned} \frac{d}{dx} [x^2 (K_1^2 - K_0^2)] &= 2x (K_1^2 - K_0^2) + x^2 \left(2K_1 \frac{dK_1}{dx} - 2K_0 \frac{dK_0}{dx} \right) \\ &= 2x (K_1^2 - K_0^2) + x^2 \left[2K_1 \left(-K_0 - \frac{K_1}{x} \right) - 2K_0 (-K_1) \right] \\ &= -2x K_0^2. \end{aligned} \quad (122)$$

Therefore, the integral is

$$\int_{\xi_{\min}}^{\infty} \left[\xi K_1^2(\xi) + \frac{\xi}{\gamma^2} K_0^2(\xi) \right] d\xi = \left[-\xi K_0 K_1 - \frac{\xi^2}{2\gamma^2} (K_1^2 - K_0^2) \right] \Big|_{\xi_{\min}}^{\infty} \quad (35')$$

$$= \xi_{\min} K_0(\xi_{\min}) K_1(\xi_{\min}) - \frac{\xi_{\min}^2}{2\gamma^2} [K_1^2(\xi_{\min}) - K_0^2(\xi_{\min})]. \quad (123)$$

For particular relativistic particles,

$$\xi_{\min} = \frac{\omega b_{\min}}{\gamma v} \lesssim 10^{-7}. \quad (124)$$

For $\xi \ll 1$,

$$K_1(\xi) \simeq \frac{1}{\xi}, \quad K_0(\xi) \simeq \ln \left(\frac{1.123}{\xi} \right). \quad (125)$$

Thus the integral becomes

$$\begin{aligned} &\simeq \ln \left(\frac{1.123}{\xi_{\min}} \right) - \frac{1}{2\gamma^2} \left[1 - \xi_{\min}^2 \ln^2 \left(\frac{1.123}{\xi_{\min}} \right) \right] \\ &\simeq \ln \left(\frac{1.123}{\xi_{\min}} \right) - \frac{1}{2\gamma^2}. \end{aligned} \quad (126)$$

Hence,

$$\frac{dE}{dx} = \frac{4\pi N Z 3^2 e^4}{mv^2} \sum_j f_j \left[\ln \left(\frac{1.123}{\xi_{\min}} \right) - \frac{v^2}{2c^2} \right]. \quad (36)$$

Define the mean frequency by

$$\ln \langle \omega \rangle = \frac{\sum_j f_j \ln \omega_j}{Z}. \quad (38)$$

Then

$$\begin{aligned} \sum_j f_j \ln \left(\frac{\alpha}{\omega_j} \right) &= \sum_j f_j \ln \alpha - \sum_j f_j \ln \omega_j \\ &= Z \ln \alpha - Z \ln \langle \omega \rangle \\ &= Z \ln \left(\frac{\alpha}{\langle \omega \rangle} \right). \end{aligned} \quad (127)$$

We obtain the classical Bohr formula (1915):

$$\frac{dE}{dx} = \frac{4\pi N Z 3^2 e^4}{mv^2} \left[\ln B_c - \frac{v^2}{2c^2} \right], \quad (36)$$

where

$$B_c = \frac{1.123 \gamma v}{2 \langle \omega \rangle b_{\min}} = \frac{1.123 \gamma^2 m v^3}{3 e^2 \langle \omega \rangle}. \quad (37)$$

This is a good approximation for heavy particles ($Z\alpha \ll 1$) and for a large number of collisions.

For the quantum correction, first note that, by simplicity, the maximum energy transfer may be approximated as

$$\Delta E(b_{\max}) \simeq \frac{1}{b_{\max}^2} 3^2 \left(\frac{v_0}{v} \right)^4 \frac{(\hbar \omega_0)^2}{I_H}, \quad (39)$$

where

$$v_0 = \frac{\hbar}{ma_0} = \left(\frac{e^2}{\hbar c} \right) c = \alpha c = \frac{c}{137} \quad (128)$$

is the orbital velocity of an electron in the fundamental state, and

$$I_H = \frac{me^4}{2\hbar^2} = 13.6 \text{ eV} \quad (129)$$

is the ionization potential of hydrogen.
For the corresponding harmonic oscillator,

$$I_H = \frac{m(ke^2)^2}{2\hbar^2} \quad (130)$$

and

$$I_T = \frac{m(\omega_0 a_0)^2}{2} = \frac{\hbar^2}{2ma_0^2} = \frac{\hbar\omega_0}{2}. \quad (131)$$

Thus, for $v \gg v_0$,

$$\Delta E \ll I_H. \quad (132)$$

After many collisions, the quantum fluctuation satisfies $\delta E_q \ll 1$, and the classical formula can be used.

The uncertainty principle gives

$$\Delta x \gtrsim \frac{\hbar}{\Delta p} \implies b_{\min} \simeq \Delta x \gtrsim \frac{\hbar}{\Delta p}, \quad (133)$$

and hence

$$(b_{\min}^q)^2 = \frac{\hbar}{2\gamma mv}. \quad (40)$$

We choose the larger of $b_{\min}^c$, Eq. (6), and $b_{\min}^q$, Eq. (40). Their ratio is

$$\frac{b_{\min}^c}{b_{\min}^q} = \frac{3e^2}{\gamma mv^2} \frac{\gamma mv}{\hbar} = \frac{3e^2}{\hbar v} \equiv \eta. \quad (42)$$

The classical formula applies for $\eta > 1$, i.e., $3e^2 \gg \hbar v$, or $v \ll 1$ in the corresponding natural-unit notation.

For $\eta < 1$, we modify the logarithm:

$$B_q = \frac{b_{\max}}{b_{\min}^q} = \eta B \simeq \frac{\gamma^2 mv^2}{\hbar \langle \omega \rangle}. \quad (43)$$

We then obtain the Bethe formula (1930):

$$\frac{dE_q}{dx} = \frac{4\pi NZ 3^2 e^4}{mv^2} \left[\ln \left(\frac{2\gamma^2 mv^2}{\hbar \langle \omega \rangle} \right) - \frac{v^2}{c^2} \right]. \quad (44)$$

For electrons incident on electrons, the calculation of the momentum in the center-of-momentum frame gives

$$(b_{\min}^q)_e = \frac{\hbar}{mc} \sqrt{\frac{2}{\gamma - 1}}. \quad (41)$$

Modifying the logarithm,

$$B_e \simeq (\gamma - 1) \sqrt{\frac{\gamma - 1}{2}} \frac{mc^2}{\hbar \langle \omega \rangle} \simeq \frac{\gamma^{3/2} mc^2}{\sqrt{2} \hbar \langle \omega \rangle}. \quad (45)$$

In the case of nonrelativistic collisions ($\Delta E \ll E \simeq 10 \text{ keV}$),

$$b_{\min}(E) = \frac{3e^2}{\gamma m v^2} = \frac{3e^2}{\gamma m v \sqrt{2mc^2 E}}, \quad (134)$$

and therefore

$$B_q(E) = \frac{\gamma m v \sqrt{2mc^2 E}}{\hbar \langle \omega \rangle}. \quad (49)$$

It is worth noting that

$$b_{\min}(E) \simeq \frac{\hbar}{\Delta p} = \frac{\hbar}{\sqrt{2mc^2 E}}. \quad (135)$$

The expressions for dE/dx obtained so far represent the mean energy loss per unit path in the medium. The quadratic mean fluctuation of the energy loss is

$$\begin{aligned} \frac{dE^2}{dx} &= 2\pi N Z \int_{b_{\min}}^{b_{\max}} [\Delta E(b)]^2 b db \\ &= \frac{4\pi N Z 3^2 e^4}{m v^2} \left[\gamma^2 m^2 v^4 \left(1 - \frac{\beta^2}{2} \right) \right], \quad b_{\min} \ll b_{\max}. \end{aligned} \quad (50)$$

The energy distribution is then given by a Gaussian:

$$P(E_0, E, x) = \frac{1}{\sqrt{\pi} \Omega} \exp \left[-\frac{(E - \bar{E})^2}{\Omega^2} \right], \quad (52)$$

where

$$\Omega^2 = x \frac{dE^2}{dx}, \quad (136)$$

E_0 is the initial energy and $\bar{E}$ is the mean final energy.

This holds for

$$\Omega \gg \Delta E_{\max} \simeq 2\gamma^2 mc^2. \quad (137)$$

For highly relativistic particles, the Gaussian is replaced by the Landau distribution.

4. Effect of Density on Energy Loss in Collisions

Consider the effect of the medium density on the energy loss. The collision with the atomic electrons is affected by the electric field of the medium itself. For an isotropic dielectric, the constitutive relation is

$$\mathbf{D}(\mathbf{x}, \omega) = \varepsilon(\omega) \mathbf{E}(\mathbf{x}, \omega). \quad (138)$$

We use the following Fourier-transform conventions:

$$\mathbf{F}(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \mathbf{F}(\mathbf{x}, \omega) e^{-i\omega t}, \quad \mathbf{F}(\mathbf{x}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \mathbf{F}(\mathbf{x}, t) e^{i\omega t}, \quad (139)$$

$$\mathbf{F}(\mathbf{x}, \omega) = \frac{1}{(\sqrt{2\pi})^3} \int d^3k \mathbf{F}(\mathbf{k}, \omega) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \mathbf{F}(\mathbf{k}, \omega) = \frac{1}{(\sqrt{2\pi})^3} \int d^3x \mathbf{F}(\mathbf{x}, \omega) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (140)$$

Combining them,

$$\mathbf{F}(\mathbf{x}, t) = \frac{1}{(2\pi)^2} \int d^3k \int d\omega \mathbf{F}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}. \quad (53)$$

Then

$$\nabla \times \mathbf{F}(\mathbf{x}, t) = i\mathbf{k} \times \mathbf{F}(\mathbf{x}, t), \quad \partial_t \mathbf{F}(\mathbf{x}, t) = -i\omega \mathbf{F}(\mathbf{x}, t). \quad (141)$$

Using

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \quad \mathbf{D} = \varepsilon \mathbf{E}, \quad (142)$$

we have

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \nabla \cdot \left(-\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \\ &= -\nabla^2 \phi - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = \frac{4\pi\rho}{\varepsilon}. \end{aligned} \quad (143)$$

Furthermore,

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu \mathbf{J} + \frac{\mu\varepsilon}{c} \frac{\partial \mathbf{E}}{\partial t} \\ &= \frac{\mu}{c} \mathbf{J} - \frac{\mu\varepsilon}{c} \nabla \frac{\partial \phi}{\partial t} - \frac{\mu\varepsilon}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}. \end{aligned} \quad (144)$$

In the Lorenz gauge,

$$\nabla \cdot \mathbf{A} + \frac{\mu\varepsilon}{c} \frac{\partial \phi}{\partial t} = 0, \quad (145)$$

and therefore

$$\nabla^2 \phi - \frac{\mu\varepsilon}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{4\pi\rho}{\varepsilon} \implies \left(k^2 - \frac{\omega^2}{c^2} \mu\varepsilon \right) \phi = \frac{4\pi\rho}{\varepsilon}, \quad (146)$$

$$\nabla^2 \mathbf{A} - \frac{\mu\varepsilon}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{J} \implies \left(k^2 - \frac{\omega^2}{c^2} \mu\varepsilon \right) \mathbf{A} = \frac{4\pi}{c} \mathbf{J}. \quad (147)$$

For a dielectric, with $\mu = 1$, the equations in Fourier space are

$$\begin{aligned} \left[k^2 - \frac{\omega^2}{c^2} \varepsilon(\omega) \right] \phi(\mathbf{k}, \omega) &= \frac{4\pi\rho(\mathbf{k}, \omega)}{\varepsilon(\omega)}, \\ \left[k^2 - \frac{\omega^2}{c^2} \varepsilon(\omega) \right] \mathbf{A}(\mathbf{k}, \omega) &= \frac{4\pi}{c} \mathbf{J}(\mathbf{k}, \omega). \end{aligned} \quad (54)$$

These are the wave equations in Fourier space.

The only source is the incident particle:

$$\rho(\mathbf{x}, t) = 3e \delta(\mathbf{x} - \mathbf{v}t), \quad \mathbf{J}(\mathbf{x}, t) = \mathbf{v}\rho(\mathbf{x}, t). \quad (55)$$

Their Fourier transforms are

$$\begin{aligned} \rho(\mathbf{k}, \omega) &= \frac{1}{(2\pi)^2} \int d^3x dt e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} 3e \delta(\mathbf{x} - \mathbf{v}t) \\ &= \frac{3e}{2\pi} \int dt e^{-i(\mathbf{k}\cdot\mathbf{v} - \omega)t} = \frac{3e}{2\pi} \delta(\omega - \mathbf{k} \cdot \mathbf{v}), \end{aligned} \quad (56)$$

$$\begin{aligned} \mathbf{J}(\mathbf{k}, \omega) &= \frac{1}{(2\pi)^2} \int d^3x dt e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \mathbf{v} 3e \delta(\mathbf{x} - \mathbf{v}t) \\ &= \mathbf{v}\rho(\mathbf{k}, \omega). \end{aligned} \quad (56')$$

Thus the Fourier transforms of the potentials are

$$\phi(\mathbf{k}, \omega) = \frac{2 \, 3e \, \delta(\omega - \mathbf{k} \cdot \mathbf{v})}{\varepsilon(\omega) [k^2 - \omega^2 \varepsilon(\omega)/c^2]}, \quad (57)$$

$$\mathbf{A}(\mathbf{k}, \omega) = \frac{4\pi}{c} \frac{\mathbf{v} \, 3e \, \delta(\omega - \mathbf{k} \cdot \mathbf{v})}{(2\pi)^2 [k^2 - \omega^2 \varepsilon(\omega)/c^2]} = \frac{\varepsilon(\omega)}{c} \mathbf{v} \phi(\mathbf{k}, \omega). \quad (148)$$

Since

$$\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (149)$$

we obtain

$$\mathbf{E}(\mathbf{k}, \omega) = i \left[\frac{\omega \varepsilon(\omega)}{c^2} \mathbf{v} - \mathbf{k} \right] \phi(\mathbf{k}, \omega), \quad (58)$$

$$\mathbf{B}(\mathbf{k}, \omega) = i \mathbf{k} \times \mathbf{A}(\mathbf{k}, \omega) = i \varepsilon(\omega) \frac{\mathbf{k} \times \mathbf{v}}{c} \phi(\mathbf{k}, \omega). \quad (58')$$

We calculate the electric field at a distance b perpendicular to the trajectory of the same particle moving in the x direction:

$$\mathbf{E}(\omega) = \frac{1}{(2\pi)^{3/2}} \int d^3 k \, \mathbf{E}(\mathbf{k}, \omega) e^{i\mathbf{k} \cdot \mathbf{b}}. \quad (59)$$

Let

$$\mathbf{E} = E_1(\omega) \hat{\mathbf{x}} + E_2(\omega) \hat{\mathbf{y}}. \quad (150)$$

Using Eqs. (57) and (58), we find

$$E_1(\omega) = \frac{2i \, 3e}{\varepsilon(\omega)(2\pi)^{3/2}} \int d^3 k \, e^{ik_2 b} \left[\frac{\omega \varepsilon(\omega) v - k_1 c^2}{c^2} \right] \frac{\delta(\omega - vk_1)}{k^2 - \omega^2 \varepsilon(\omega)/c^2}. \quad (60)$$

Integrating over k_1 ,

$$\delta(\omega - vk_1) \implies k_1 = \frac{\omega}{v}, \quad \lambda^2 = k_1^2 - \frac{\omega^2}{c^2} \varepsilon = \frac{\omega^2}{v^2} (1 - \beta^2 \varepsilon), \quad (151)$$

where $\beta = v/c$. Hence

$$E_1(\omega) = -\frac{2i \, 3e\omega}{(2\pi)^{3/2} v^2} \left[\frac{1}{\varepsilon(\omega)} - \beta^2 \right] \int_{-\infty}^{+\infty} dk_2 \, e^{ik_2 b} \int_{-\infty}^{+\infty} \frac{dk_3}{k_2^2 + k_3^2 + \lambda^2}. \quad (61)$$

Using

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{a}, \quad (152)$$

we obtain

$$E_1(\omega) = -\frac{i \, 3e\omega}{\sqrt{2\pi} \, v^2} \left[\frac{1}{\varepsilon(\omega)} - \beta^2 \right] \int_{-\infty}^{+\infty} \frac{e^{ik_2 b}}{\sqrt{\lambda^2 + k_2^2}} dk_2. \quad (62)$$

Introducing the modified Bessel function K_0 ,

$$K_0(z) = \frac{2^\nu \Gamma(\nu + 1/2)}{\sqrt{\pi}} \int_0^\infty \frac{\cos(zx)}{(1+x^2)^{\nu+1/2}} dx, \quad (153)$$

with $\nu = 0$, we have

$$K_0(z) = \int_0^\infty \frac{\cos(zx)}{\sqrt{1+x^2}} dx. \quad (154)$$

Therefore,

$$E_1(\omega) = -i \frac{3e\omega}{v^2} \left(\frac{2}{\pi}\right)^{1/2} \left[\frac{1}{\varepsilon(\omega)} - \beta^2 \right] K_0(\lambda b). \quad (63)$$

Analogously,

$$E_2(\omega) = \frac{3e}{v} \left(\frac{2}{\pi}\right)^{1/2} \frac{\lambda}{\varepsilon(\omega)} K_1(\lambda b), \quad (64)$$

and

$$B_3(\omega) = \varepsilon(\omega) \beta E_2(\omega). \quad (155)$$

In the limit $\varepsilon(\omega) \rightarrow 1$, these expressions tend to the vacuum results obtained previously. From Eq. (34),

$$\mathbf{P} = \frac{1}{4\pi} (\mathbf{D} - \mathbf{E}) = \frac{\varepsilon - 1}{4\pi} \mathbf{E}, \quad (156)$$

and

$$\mathbf{P} = -Ne \sum_j f_j \mathbf{x}_j, \quad (157)$$

where N is the number of atoms per unit volume and f_j is the fraction of atoms with amplitude x_j of the displacement. Thus,

$$\Delta E(b) = 2e \sum_j f_j \operatorname{Re} \int_0^\infty \omega x_j(\omega) E^*(\omega) d\omega. \quad (158)$$

Using the dielectric term,

$$\Delta E(b) = \frac{1}{2\pi N} \operatorname{Re} \int_0^\infty -i\omega \varepsilon(\omega) |\mathbf{E}(\omega)|^2 d\omega. \quad (65)$$

Therefore, for collisions with $b > a$,

$$\left(\frac{dE}{dx} \right)_{b>a} = 2\pi N \int_a^\infty \Delta E(b) b db. \quad (159)$$

In the geometry,

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \cdot \hat{\mathbf{r}}, \quad (160)$$

and the radial component is

$$S_r = -\frac{c}{4\pi} E_1 B_3. \quad (161)$$

Integrating over the cylindrical surface and over time,

$$\left(\frac{dE}{dx} \right) = \frac{c}{4\pi} 2\pi b \int_{-\infty}^{+\infty} dt (\mathbf{E} \times \mathbf{B}) \cdot \hat{\mathbf{r}}. \quad (162)$$

Consequently,

$$\begin{aligned}
\left(\frac{dE}{dx}\right) &= -\frac{cb}{2} \int_{-\infty}^{+\infty} dt B_3(t) E_1(t) \\
&= -\frac{cb}{2} \int d\omega d\omega' B_3(\omega') E_1(\omega) e^{-i(\omega+\omega')t} \\
&= -\frac{cb}{2} \int d\omega B_3(-\omega) E_1(\omega) \\
&= -\frac{cb}{2} \int_{-\infty}^{+\infty} d\omega E_1(\omega) B_3^*(\omega) \\
&= -cb \operatorname{Re} \int_0^\infty d\omega E_1(\omega) B_3^*(\omega).
\end{aligned} \tag{163}$$

Equivalently,

$$\mathbf{S} = \frac{c}{4\pi} \operatorname{Re} [\mathbf{E} \times \mathbf{B}^*]. \tag{164}$$

Substituting $E_1(\omega)$ and $B_3^*(\omega)$,

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{2}{\pi} \left(\frac{3e}{v}\right)^2 \operatorname{Re} \int_0^\infty i\omega \lambda^2 K_1(\lambda b) K_0(\lambda b) \frac{1-\beta^2}{\varepsilon(\omega)} d\omega. \tag{67}$$

Now consider the case of complex $\varepsilon(\omega)$. The polarization is given by

$$\mathbf{D}(\mathbf{x}, \omega) = \mathbf{E}(\mathbf{x}, \omega) + 4\pi\mathbf{P}(\mathbf{x}, \omega), \tag{165}$$

with, in the linear approximation,

$$\mathbf{P}(\omega) = \frac{ne^2}{m} \frac{\mathbf{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\Gamma}, \tag{166}$$

where n is the electronic density. The corresponding dielectric constant is

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}, \tag{167}$$

where

$$\omega_p = \sqrt{\frac{4\pi ne^2}{m}} \tag{168}$$

is the plasma frequency.

If f_j is the fraction of atoms with frequency ω_j ,

$$\varepsilon(\omega) = 1 + \frac{4\pi Ne^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\Gamma_j}. \tag{69}$$

The expressions are complicated to integrate directly. For the ultrarelativistic case ($\beta \simeq 1$), and for optical frequencies (where they are important) compared with the atomic radius,

$$|\lambda a| \simeq \frac{\omega a}{c} \ll 1. \tag{169}$$

Using the small-argument behavior of the Bessel functions,

$$\lambda^* a K_1(\lambda^* a) \simeq 1, \quad (170)$$

and

$$K_0(\lambda a) \simeq \ln \left(\frac{1.123}{\lambda a} \right), \quad (171)$$

the stopping expression becomes

$$\left(\frac{dE}{dx} \right)_{b>a} \simeq \frac{2}{\pi} \left(\frac{3e}{c} \right)^2 \operatorname{Re} \int_0^\infty i\omega \ln \left(\frac{1.123}{\lambda a} \right) \left[\frac{1}{\varepsilon(\omega)} - \beta^2 \right] d\omega. \quad (172)$$

In the limit $\beta \rightarrow 1$ ($v \rightarrow c$), define

$$\lambda \simeq \frac{\omega}{c} (1 - \beta^2 \varepsilon)^{1/2}, \quad (173)$$

so that

$$\ln \left(\frac{1.123}{\lambda a} \right) \simeq \ln \left(\frac{1.123c}{\omega a} \right) - \frac{1}{2} \ln(1 - \varepsilon). \quad (174)$$

Thus,

$$\left(\frac{dE}{dx} \right)_{b>a} \simeq \frac{2}{\pi} \left(\frac{3e}{c} \right)^2 \operatorname{Re} \int_0^\infty i\omega \left(\frac{1}{\varepsilon(\omega)} - 1 \right) \left[\ln \left(\frac{1.123c}{\omega a} \right) - \frac{1}{2} \ln(1 - \varepsilon(\omega)) \right] d\omega. \quad (70)$$

Define

$$\left(\frac{dE}{dx} \right)_{b>a} \simeq \frac{2}{\pi} \left(\frac{3e}{c} \right)^2 \operatorname{Re}[I], \quad (175)$$

where

$$I = \int_0^\infty i\omega \left[\ln \left(\frac{1.123c}{\omega_p a} \right) - \ln \omega + \frac{1}{2} \ln(\omega^2 - \omega_0^2 + i\omega\Gamma) \right] \frac{-\omega_p^2}{\omega_0^2 + \omega_p^2 - \omega^2 - i\omega\Gamma} d\omega. \quad (176)$$

This integral may be evaluated by Cauchy's theorem, closing the contour in the upper half-plane. The contour contains no poles in its interior, since the sum of the poles is zero. The contribution from the imaginary-axis contour is obtained by setting $\omega = i\Omega$, with Ω real. One finds

$$\begin{aligned} I_3 &= i \int_0^\infty \Omega \left[\ln \left(\frac{1.123c}{\omega_p a} \right) - \ln \Omega + \frac{1}{2} \ln(\Omega^2 + \omega_0^2 + \Omega\Gamma) \right] \\ &\quad \times \frac{\omega_p^2}{\Omega^2 + \omega_0^2 + \omega_p^2 + \Omega\Gamma} d\Omega, \end{aligned} \quad (177)$$

and

$$\operatorname{Re}(I_3) = 0. \quad (178)$$

For the large semicircle, let $\omega = \Omega e^{i\theta}$ with $\Omega \rightarrow \infty$ and $\theta \in [0, \pi/2]$. The logarithmic factor vanishes asymptotically, so that the corresponding contribution tends to zero.

Thus the integral along the real axis is determined by the remaining part of the contour. The result is

$$I = -\frac{\pi\omega_p^2}{2} \ln \left(\frac{1.123c}{\omega_p a} \right). \quad (179)$$

Therefore,

$$\left(\frac{dE}{dx}\right)_{b>a} = \left(\frac{3e\omega_p}{c}\right)^2 \ln\left(\frac{1.123c}{\omega_p a}\right). \quad (71)$$

Comparing with Eq. (36), and using

$$\omega_p^2 = \frac{4\pi N Z e^2}{m}, \quad (72)$$

together with

$$B_c = \frac{1.123\gamma c}{a\omega}, \quad (180)$$

we obtain

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{(3e)^2\omega_p^2}{c^2} \left[\ln\left(\frac{1.123\gamma c}{a\langle\omega\rangle}\right) - \frac{1}{2} \right]. \quad (73)$$

For the final formula for ionization, one takes

$$\langle\omega\rangle \longrightarrow \omega_p, \quad f \longrightarrow 1 \quad (\text{independent}). \quad (181)$$

It is important to note a reduction in the energy loss at high energies: the density effect modifies the asymptotic behavior of dE/dx .

Chapter 14 – Radiation from Moving Charges

1. Liénard–Wiechert Potentials

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