Símbolos de Christoffel a partir das equações de Euler-Lagrange

$$L = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \quad \text{[veja eq. 3.3.14 do Wald]}$$

* Tomemos como exemplo a métrica $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 + r^2\left(d\theta^2 + \operatorname{sen}^2\theta\, d\phi^2\right)$

$$\Rightarrow L = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1+\frac{2M}{r}\right)\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 + r^2\operatorname{sen}^2\theta\left(\frac{d\phi}{d\lambda}\right)^2$$

* Equações de Euler Lagrange: $\dfrac{\partial L}{\partial x^\mu(\lambda)} - \dfrac{d}{d\lambda}\left(\dfrac{\partial L}{\partial x'^\mu(\lambda)}\right) = 0$

Por exemplo, para $\mu = 0$ temos $\dfrac{\partial L}{\partial t} - \dfrac{d}{d\lambda}\left(\dfrac{\partial L}{\partial t'}\right) = 0$

Como L não depende explicitamente de t, temos que $\frac{\partial L}{\partial t} = 0$. Por outro lado,

$$\frac{\partial L}{\partial t'} = \frac{\partial L}{\partial\left(\frac{dt}{d\lambda}\right)} = -\left(1-\frac{2M}{r}\right)\cdot 2\cdot\left(\frac{dt}{d\lambda}\right) \Rightarrow \frac{d}{d\lambda}\left(\frac{\partial L}{\partial t'}\right) = -2\frac{d}{d\lambda}\left[\left(1-\frac{2M}{r}\right)\cdot\frac{dt}{d\lambda}\right]$$

$$= -2\cdot\frac{2M}{r^2}\frac{dr}{d\lambda}\cdot\frac{dt}{d\lambda} - 2\cdot\left(1-\frac{2M}{r}\right)\frac{d^2t}{d\lambda^2}$$

repetir o mesmo para $\mu=1$, $\mu=2$ e $\mu=3$.

Ou seja: $\dfrac{\partial L}{\partial t} - \dfrac{d}{d\lambda}\left(\dfrac{\partial L}{\partial t'}\right) = 0 \Rightarrow -2\cdot\dfrac{2M}{r^2}\cdot\dfrac{dr}{d\lambda}\cdot\dfrac{dt}{d\lambda} - 2\cdot\left(1-\dfrac{2M}{r}\right)\cdot\dfrac{d^2t}{d\lambda^2} = 0$

$$\Rightarrow \frac{d^2t}{d\lambda^2} + \frac{\frac{2M}{r^2}}{1-\frac{2M}{r}}\cdot\frac{dr}{d\lambda}\cdot\frac{dt}{d\lambda} = 0 \Rightarrow \boxed{\frac{d^2t}{d\lambda^2} + \frac{2M}{r^2-2Mr}\cdot\frac{dr}{d\lambda}\cdot\frac{dt}{d\lambda} = 0}$$

essa é a componente 0 da equação geodésica $\dfrac{d^2x^0}{d\lambda^2} + \Gamma^0_{\mu\nu}\dfrac{dx^\mu}{d\lambda}\dfrac{dx^\nu}{d\lambda} = 0$

Comparando com $\dfrac{d^2x^0}{d\lambda^2} + \Gamma^0_{\mu\nu}\dfrac{dx^\mu}{d\lambda}\dfrac{dx^\nu}{d\lambda} = 0 = \dfrac{d^2t}{d\lambda^2} + \Gamma^0_{00}\left(\dfrac{dt}{d\lambda}\right)^2 + \Gamma^0_{11}\left(\dfrac{dr}{d\lambda}\right)^2 + \Gamma^0_{22}\left(\dfrac{d\theta}{d\lambda}\right)^2 + \Gamma^0_{33}\left(\dfrac{d\phi}{d\lambda}\right)^2 + 2\Gamma^0_{01}\dfrac{dt}{d\lambda}\dfrac{dr}{d\lambda}$

$+ 2\Gamma^0_{02}\dfrac{dt}{d\lambda}\dfrac{d\theta}{d\lambda} + 2\Gamma^0_{03}\dfrac{dt}{d\lambda}\dfrac{d\phi}{d\lambda} + 2\Gamma^0_{12}\dfrac{dr}{d\lambda}\dfrac{d\theta}{d\lambda} + 2\Gamma^0_{13}\dfrac{dr}{d\lambda}\dfrac{d\phi}{d\lambda} + 2\Gamma^0_{23}\dfrac{d\theta}{d\lambda}\dfrac{d\phi}{d\lambda} \Rightarrow \Gamma^0_{00} = \Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = \Gamma^0_{02} = \Gamma^0_{03} = \Gamma^0_{12} = \Gamma^0_{13} = \Gamma^0_{23} = 0$ e $\boxed{\Gamma^0_{01} = \dfrac{M}{r^2-2Mr}}$