CÁLCULO VETORIAL E TENSORIAL - LISTA 1

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1. Dados vetores $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^3$ e constantes $a, b \in \mathbb{R}$, mostre que:

   (a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, e $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

   (b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

   (c) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

   (d) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = (uv)^2 - (\mathbf{u} \cdot \mathbf{v})^2$, onde $u = \|\mathbf{u}\|$ e $v = \|\mathbf{v}\|$

   (e) $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = u^2 - v^2$

   (f) $(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v}) = 2\mathbf{u} \times \mathbf{v}$

   (g) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v})$

   (h) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}) - (\mathbf{u} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{w})$

   (i) (identidade de Jacobi) $\mathbf{u} \times \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} \times \mathbf{u} = 0$

2. Desenhe as superfícies de nível correspondentes aos respectivos valores $c = 0, 1, 4, 9$ para a função $f(x, y) = x^2 + y^2$

3. Calcule $\frac{\partial^2 f}{\partial x \partial y}$ dos seguintes campos escalares:
   a) $f(x, y) = \ln(x^2 + y^2)$, $(x, y) \neq (0, 0)$, b) $f(x, y) = \text{atan}(y/x)$, $x \neq 0$

4. Seja $v(r, t) = t^n \exp(-\frac{r^2}{4t})$. Calcule n para que o campo escalar v satisfaça a equação $\frac{\partial v}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right)$.

5. Calcule o gradiente ∇f dos seguintes campos escalares: a) $f(x, y) = x^2 + y \cos(xy)$, b) $f(x, y) = e^{x^2} \sin y$, c) $\ln(x^2 + 2y^3 - z^4)$
   Calcule também $\|\nabla f\|$ para cada item acima.

6. Considere duas funções $g : \mathbb{R}^3 \to \mathbb{R}$ e $f : \mathbb{R}^3 \to \mathbb{R}$. Prove que $\nabla(fg) = f\nabla g + g\nabla f$.

7. Mostre que $\nabla(r^n) = nr^{n-2}\mathbf{r}$

8. Mostre que os operadores divergente e rotacional são *lineares*, ou seja, dados $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ e constantes $a, b \in \mathbb{R}$ segue-se que:
   a) $\nabla \cdot (a\mathbf{u} + b\mathbf{v}) = a\nabla \cdot \mathbf{u} + b\nabla \cdot \mathbf{v}$
   b) $\nabla \times (a\mathbf{u} + b\mathbf{v}) = a\nabla \times \mathbf{u} + b\nabla \times \mathbf{v}$

9. Mostre que

   (a) $\nabla \times (f\mathbf{u}) = f\nabla \times \mathbf{u} + (\nabla f) \times \mathbf{u}$

   (b) $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$

10. Calcule $\nabla \cdot \mathbf{u}$ e $\nabla \times \mathbf{u}$ para os seguintes campos vetoriais:

    (a) $\mathbf{u}(x, y, z) = (x^3 + yz^2)\hat{\imath} + (y + xz)\hat{\jmath} + (z + xy)\hat{k}$

    (b) $\mathbf{u}(x, y, z) = (z - 3y^2)\hat{\imath} + (3x - z)\hat{\jmath} + (y - 2x)\hat{k}$

    (c) $\mathbf{u}(x, y, z) = \exp(xy^2)\hat{\imath} + \cos(x^2 y)\hat{\jmath} + \cos(x^2 z^2)\hat{k}$