

# CÁLCULO VETORIAL E TENSORIAL - LISTA 1

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1. Dados vetores  $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^3$  e constantes  $a, b \in \mathbb{R}$ , mostre que:
  - (a)  $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ , e  $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
  - (b)  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  - (c)  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  - (d)  $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = (uv)^2 - (\mathbf{u} \cdot \mathbf{v})^2$ , onde  $u = \|\mathbf{u}\|$  e  $v = \|\mathbf{v}\|$
  - (e)  $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = u^2 - v^2$
  - (f)  $(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v}) = 2\mathbf{u} \times \mathbf{v}$
  - (g)  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v})$
  - (h)  $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}) - (\mathbf{u} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{w})$
  - (i) (identidade de Jacobi)  $\mathbf{u} \times \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} \times \mathbf{u} = 0$
2. Desenhe as superfícies de nível correspondentes aos respectivos valores  $c = 0, 1, 4, 9$  para a função  $f(x, y) = x^2 + y^2$
3. Calcule  $\frac{\partial^2 f}{\partial x \partial y}$  dos seguintes campos escalares:
  - a)  $f(x, y) = \ln(x^2 + y^2)$ ,  $(x, y) \neq (0, 0)$ ,
  - b)  $f(x, y) = \text{atan}(y/x)$ ,  $x \neq 0$
4. Seja  $v(r, t) = t^n \exp(-\frac{r^2}{4t})$ . Calcule  $n$  para que o campo escalar  $v$  satisfaça a equação  $\frac{\partial v}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v}{\partial r} \right)$ .
5. Calcule o gradiente  $\nabla f$  dos seguintes campos escalares: a)  $f(x, y) = x^2 + y \cos(xy)$ , b)  $f(x, y) = e^{x^2} \sin y$ , c)  $\ln(x^2 + 2y^3 - z^4)$   
Calcule também  $\|\nabla f\|$  para cada item acima.
6. Considere duas funções  $g : \mathbb{R}^3 \rightarrow \mathbb{R}$  e  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ . Prove que  $\nabla(fg) = f\nabla g + g\nabla f$ .
7. Mostre que  $\nabla(r^n) = nr^{n-2}\mathbf{r}$
8. Mostre que os operadores divergente e rotacional são *lineares*, ou seja, dados  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$  e constantes  $a, b \in \mathbb{R}$  segue-se que:
  - a)  $\nabla \cdot (a\mathbf{u} + b\mathbf{v}) = a\nabla \cdot \mathbf{u} + b\nabla \cdot \mathbf{v}$
  - b)  $\nabla \times (a\mathbf{u} + b\mathbf{v}) = a\nabla \times \mathbf{u} + b\nabla \times \mathbf{v}$
9. Mostre que
  - (a)  $\nabla \times (f\mathbf{u}) = f\nabla \times \mathbf{u} + (\nabla f) \times \mathbf{u}$
  - (b)  $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$
10. Calcule  $\nabla \cdot \mathbf{u}$  e  $\nabla \times \mathbf{u}$  para os seguintes campos vetoriais:
  - (a)  $\mathbf{u}(x, y, z) = (x^3 + yz^2)\hat{i} + (y + xz)\hat{j} + (z + xy)\hat{k}$
  - (b)  $\mathbf{u}(x, y, z) = (z - 3y^2)\hat{i} + (3x - z)\hat{j} + (y - 2x)\hat{k}$
  - (c)  $\mathbf{u}(x, y, z) = \exp(xy^2)\hat{i} + \cos(x^2y)\hat{j} + \cos(x^2z^2)\hat{k}$
11. Dados vetores  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$  e constantes  $a, b \in \mathbb{R}$ , mostre que:
  - (a)  $\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}$ .
  - (b)  $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$ .