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AN EQUATION UNIFYING BOTH CAMASSA-HOLM AND NOVIKOV EQUATIONS

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Abstract. In this paper we derive a new equation unifying the Camassa-Holm and Novikov equations invariant under the scaling transformation $(x, t, u) \mapsto (x, \lambda^{-b}t, \lambda u)$ and admitting a certain multiplier.

1. **Introduction.** This work corresponds to a talk given by the first author in the session *SS 69: Lie Symmetries, Conservation laws and other approaches in solving nonlinear differential equations*, organized by Chaudry Masood Khalique, Maria Gandarias and Mufid Abudiab, and also another talk, presented during the Student Paper Competition, in the 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications, which took place in Madrid, Spain, from July 07th until July 11th, 2014. We would like to thank the organizers of the session SS 69 for their kind invitation and also the organizers of the Student Paper Competition for the opportunity given to P. L. da Silva discuss our results [12].

In this communication we give a new proof of the results obtained in [12, 13] and in some parts the presentation closely follows those references. Actually, a considerable and important part of this paper was influenced by the fruitful discussions that we had with S. Anco during the event and then, we are pleased in reporting this new deduction in the Proceedings of the conference.

Since the celebrated Korteweg and de Vries paper [28], in which a third order evolution equation was derived and named after them, a huge number of papers in the literature has been done for modeling, or related with, shallow water equations. During the last century, a sequence of papers, starting with [30], showed and enlightened many properties of such equation. Additionally, the KdV equation

$$u_t = u_{xxx} + uu_x \tag{1}$$

proved to be a prototype equation for many phenomena, see, for instance, [1].

Although its good and versatile properties, the equation was not above criticisms. In the seminal paper [5], the authors derived a new equation for moderately long waves of small amplitudes whose formal justification is as that for the KdV and from that paper arose the well know Benjamin-Bona-Mahoney (BBM) equation

$$u_t = u_{txx} + uu_x. \tag{2}$$

However, the differences between both equations are greater than the fact that (1) is an evolution equation whereas (2) is not. In [5] the authors found three conserved quantities

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on the solutions of (2). Later, in [33], those obtained conservation laws were proved to be the only three admitted by (2). This fact shows a dramatic difference between the two equations since (1) admits an infinite number of conserved quantities [31].

Camassa and Holm [10], using Hamiltonian methods, derived the famous Camassa-Holm (CH) equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{3}$$

This last equation possesses remarkable properties such as solutions with peaks in which have discontinuous first order derivatives, called *peakon* solutions, and it has a bi-hamiltoninan structure, see [10], which implies in the existence of an infinite number of conserved quantities, like the KdV equation [19, 31, 29]. Moreover, it admits Lie symmetries generated by the differential operators

$$X = \frac{\partial}{\partial x}, \quad T = \frac{\partial}{\partial t} \tag{4}$$

and

$$X_1 = u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t}. \tag{5}$$

Since Camassa and Holm's work much interest have been paid to third order equations having similar properties as those known for KdV and CH equations. For instance, in [15] it was derived an integrable equation having peakon solutions with first order nonlinearities, while in [16] another integrable one, combining linear effects in the dispersion, such as in the KdV case, and nonlinear dispersion, like the CH equation, was reported. More recently, Novikov [32] has discovered the equation

$$u_t - u_{txx} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \tag{6}$$

which not only admits peakon solutions and has cubic nonlinearities, but it is also integrable [18].

In [23], Ibragimov, Khamitova and Valenti, using the techniques introduced in [22], showed that (3) is strictly self-adjoint and, from the Lie point symmetry generator (5), they established the following conserved vector for (3):

$$C^0 = u^2 + u_x^2, \quad C^1 = 2(u^2 - u^2 u_{xx} - uu_{tx}). \tag{7}$$

One of us (I. L. Freire), jointly with Y. Bozhkov and N. Ibragimov, considered in [9] the Novikov equation (6) from the point of view of Lie group analysis. It was shown that equation (6) admits a five dimensional symmetry Lie algebra, spanned by the operators (4),

$$X_+ = e^{2x}\frac{\partial}{\partial x} + e^{2x}u\frac{\partial}{\partial u}, \qquad X_- = e^{-2x}\frac{\partial}{\partial x} - e^{-2x}u\frac{\partial}{\partial u}$$

and

$$X_2 = -2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \tag{8}$$

In the same work, and using again the same techniques [22], it was obtained the following conserved vector from the generator (8):

$$C^0 = u^2 + u_x^2, \quad C^1 = 2u^4 - 2u^3 u_{xx} - 2uu_{tx}. \tag{9}$$

We recall that the components C^0 in (7) and (9) are the conserved quantities employed for constructing a Hamiltonian for (3) and (6), see [10] and [18], respectively, for the Camassa-Holm and Novikov equations.

More recently, in [14] we considered the modified Novikov equation

$$u_t - u_{txx} + (b+1)u^2 u_x - buu_x u_{xx} - u^2 u_{xxx} = 0, \tag{10}$$

introduced in [27]. We showed that for any $b \neq 3$, the Lie point symmetries of (10) only admits the generators (4) and (8). In the case $b = 3$, we recover the Novikov equation and, for this case, in addition to (4) and (8), we have the additional generators X_+ and X_- mentioned above.

Investigating the construction of conservation laws using Ibragimov's ideas [22], we were surprised in concluding that a local conservation law for (10), using the approach [22, 24, 25], could only be obtained if and only if $b = 3$, that is, only when it is considered the Novikov equation.

Therefore, motivated by these recent works, we investigated [12, 13] the following class of third order equation

$$u_t + \varepsilon u_{txx} + f(u)u_x + g(u)u_x u_{xx} + h(u)u_{xxx} = 0. \tag{11}$$

However, differently from [12], in this communication we consider (11) not from the point of view of strict self-adjointness [22, 25], but we now move our eyes to the direct method [2, 3, 4, 6].

The paper is organized as the follows. In the next section we present some basic facts on Lie symmetries. Conservation laws are revisited in the next. The main idea of the direct method is presented in section 4. Then, in the section 5, we present a new equation connecting Camassa-Holm and Novikov equations.

2. **Symmetries.** Here we present a very short recall in Lie symmetries. For further details, see [7, 8, 20, 21, 36].

Let $x = (x^1, \cdots, x^n) \in X \subseteq \mathbb{R}^n$, $u = u(x,t) \in U \subseteq \mathbb{R}$ and $u_{(j)}$ be, respectively, the set of the independent and dependent variables; and the set of all $j-th$ derivatives of u. Hereafter, the summation over repeated indices is presupposed. All functions here are assumed to be smooth. In particular, $u_{i_1\cdots i_j} = D_{i_1}\cdots D_{i_j}(u)$, where

$$D_i := \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots, \quad i = 0, \cdots, n$$

are the total derivative operators.

Let $\mathcal{A}$ be the set of all locally analytic functions of a finite number of the variables x, u and $u_{(j)}$. Let $F \in \mathcal{A}$ and consider an equation

$$F(x, u, u_{(1)}, \cdots, u_{(k)}) = 0. \tag{12}$$

An operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u} \tag{13}$$

is called Lie point symmetry generator of the equation (12) if

$$X^{(k)}F = \lambda F, \tag{14}$$

for a certain function λ depending on $x, u, u_{(1)}, \cdots$. In this case, the Lie point symmetry is given by $(x,u) \mapsto (\bar{x}, \bar{u})$, where $x = (x^1, \cdots, x^n)$ and

$$\begin{aligned} \bar{x}^i &= x^i + \varepsilon X x^i + \frac{\varepsilon^2}{2!}X(Xx^i) + \cdots + \frac{\varepsilon^n}{n!}X(X^{n-1}x^i) + \cdots = x^i + \sum_{j=1}^{\infty}\frac{\varepsilon^j}{j!}X^j(x^i), \\ \bar{u} &= u + \varepsilon X u + \frac{\varepsilon^2}{2!}X(Xu) + \cdots + \frac{\varepsilon^n}{n!}X(X^{n-1}u) + \cdots = u + \sum_{j=1}^{\infty}\frac{\varepsilon^j}{j!}X^j(u). \end{aligned}$$

Equation (14) is called invariance condition and

$$X^{(k)} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u} + \zeta_i\frac{\partial}{\partial u_i} + \zeta_{ij}\frac{\partial}{\partial u_{ij}} + \cdots + \zeta_{i_1\cdots i_k}\frac{\partial}{\partial u_{i_1\cdots i_k}}, \tag{15}$$

where $\zeta_i = D_i\eta - D_i(\xi^j)u_j, \cdots, \zeta_{i_1\cdots i_k} = D_{i_1}\cdots D_{i_1}\eta - D_{i_k}(\xi^j)u_{i_1\cdots i_{k-1}j}$, is the $k-th$ prolongation of the vector field X. In this case, we say that $(x,u) \mapsto (\bar{x},\bar{u})$ is a Lie point symmetry of (12). Symmetries of some equations of the type (11) can be found in [11].

3. **Conservation laws.** Here we present some elements regarding conservation laws. However, the interested reader is refereed to [2, 22, 23, 25, 33, 34, 35, 40] for further details. We also guide the curious reader to [21, 26, 36, 37, 38, 39] for additional readings.

Mathematically speaking, one can define a conservation law for (12) starting from the expression

$$Div(C) := D_t C^0 + D_i C^i = \lambda F, \tag{16}$$

for a certain vector field $C := (C^0, C_x)$, where $C_x := (C^1, \cdots, C^n)$, and functions $\lambda = \lambda(t, x, u, \cdots)$. Equation (16) is called characteristic form of the conservation law $D_t C^0 + D_i C^i = 0$, while λ is its characteristic. Then, on the solutions of $F = 0$, we have $Div(C) \equiv 0$.

A vector field $C = (C^0, C_x)$ provides a *trivial conservation laws* if $Div(C) \equiv 0$. Such a vector C is, therefore, called *trivial conserved vector*. Otherwise C is called *nontrivial* conserved vector. Two conserved vectors are said to be *equivalent* if they differ by a trivial conserved vector. Clearly two equivalent conserved vectors possess the same characteristic λ. A conservation law of an equation can now be rigorously defined as follows.

By *conservation law* of (12) we mean the equivalence class of conserved vectors of (12). Then, the set of all conservation laws is a vector space whose the identity is the equivalence class of the trivial conserved quantities.

On (12), the relation (16) becomes $D_t C^0 + D_i C^i \equiv 0$. From the physical point of view, the vector field $C = (C^0, C^1, \cdots, C^n)$ is usually a *density* and it is called *conserved vector or conserved current* of the phenomena modeled by (12). The component C^0 is the conserved density while the remaining components are the conserved flux. Being a density, restricting x to a fixed domain $\Omega \subseteq \mathbb{R}^n$, with a smooth, constant, boundary $\partial\Omega$, and defining

$$Q_\Omega := \int_\Omega C^0 dx,$$

application of the divergence theorem gives

$$\frac{dQ_\Omega}{dt} = \int_\Omega D_t C^0 dx = -\int_\Omega D_i C^i dx = -\int_{\partial\Omega} C_x \cdot dS.$$

Therefore, restricted to Ω, the quantity Q_Ω depends only on the behavior of the solutions on the boundary $\partial\Omega$ and it is equal to the total flux over it. For non-dissipative physical models, this fact provides the general form of a conservation law.

4. **Direct method.** Usually, once the invariance group of an Euler-Lagrange equation is known, the celebrated Noether theorem can be invoked for establishing conservation laws for the equation under consideration. However, in our case, equations of the type (11) are not Euler-Lagrange equations and a direct application of Noether's approach is impossible.

In [22, 25] Ibragimov introduced new techniques for overcoming this problem and in [12] we applied those ideas for establishing conservation laws for equations of the type (11). However, as we pointed out in the Introduction, in this communication we shall use the direct method [2, 3, 4, 6] for obtaining the conserved currents.

The main idea is to use the identity

$$\frac{\delta}{\delta u} D_i \equiv 0,$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + \cdots + (-1)^k D_{i_1} \cdots D_{i_k} \frac{\partial}{\partial u_{i_1 \cdots i_k}} + \cdots$$

is the Euler-Lagrange operator.

Then, given a differential equation (12), our purpose is to find a function $\mu = \mu(x, u, u_{(1)}, \cdots)$, called *multiplier*, such that

$$\frac{\delta}{\delta u}(\mu F) \equiv 0. \tag{17}$$

This last result implies, once μ is found, that $\mu F = D_i C^i$ and then, on the solutions of $F = 0$ the vector field $C = (C^1, \cdots, C^n)$ provides a conservation law for (12). For further details, see [2, 3, 4, 6].

5. **A new equation unifying Camassa-Holm and Novikov equations.** Let us now obtain a one-parameter equation unifying both Camassa-Holm (3) and Novikov (6) equations. In what follows, we present a different deduction from the original one obtained in [12].

We firstly impose that the operator

$$X_b = u\frac{\partial}{\partial u} - bt\frac{\partial}{\partial t} \tag{18}$$

is a Lie point symmetry of (11). The corresponding transformation is given by

$$(x, t, u) \mapsto (x, \lambda^{-b}t, \lambda u), \tag{19}$$

where $\lambda > 0$. Then, substituting (19) into (11), one obtains

$$u_t + \varepsilon u_{txx} + \lambda^{-b}[f(\lambda u)u_x + \lambda g(\lambda u)u_x u_{xx} + h(\lambda u)u_{xxx}] = 0,$$

which should hold for any $\lambda > 0$. Then, differentiating the last equation with respect to λ, we obtain

$$\begin{aligned}&[-bf(\lambda u) + \lambda u f'(\lambda u)]u_x + [-b\lambda g(\lambda u) + \lambda^2 u g'(\lambda u)]u_x u_{xx}\\&+[-bh(\lambda u) + \lambda u h'(\lambda u)]u u_{xxx} = 0.\end{aligned} \tag{20}$$

Equation (20) is an identity for any $\lambda > 0$. From it, we have the following system of ordinary differential equations

$$\frac{d}{dz}(z^{-b}f(z)) = 0, \qquad \frac{d}{dz}(z^{1-b}g(z)) = 0, \qquad \frac{d}{dz}(z^{-b}h(z)) = 0,$$

under the change $z = \lambda u$.

The solution of the last system is given by $f(u) = \gamma u^b$, $g(u) = \sigma u^{b-1}$ and $h(u) = \delta u^b$, where γ, σ and δ are arbitrary constants. Substituting these functions into (11), we prove the following result.

Theorem 5.1. *Equation* (11) *admits the symmetry* (19) *iff it takes the form*

$$u_t + \varepsilon u_{txx} + \gamma u^b u_x + \sigma u^{b-1}u_x u_{xx} + \delta u^b u_{xxx} = 0, \tag{21}$$

where γ, σ and δ are arbitrary constants.

Our next result is:

Theorem 5.2. *Equation* (21) *admits a conserved quantity*

$$H = \int_{\mathbb{R}} (u^2 - \varepsilon u_x^2)dx \tag{22}$$

if and only if $\sigma = (b+1)\delta$.

Remark 1. We observe that to prove Theorem 5.2 it is enough to find a vector field $C = (C^0, C^1)$ such that

$$D_t C^0 + D_x C^1 = 0 \tag{23}$$

on the solutions of (20) and $C^0 = u^2 - \varepsilon u_x^2$.

Proof. Substituting $F = u_t + \varepsilon u_{txx} + \gamma u^b u_x + \sigma u^{b-1}u_x u_{xx} + \delta u^b u_{xxx}$ into (17) and taking $\mu = u$, the left side gives

$$\frac{\delta}{\delta u}(uF) \quad = \quad [\sigma - (b+1)\delta](b-1)bu^{b-2}u_x^3 + 3b[\sigma - (b+1)\delta]u^{b-1}u_x u_{xx}. \tag{24}$$

Then, we conclude that

$$\frac{\delta}{\delta u}[u(u_t + \varepsilon u_{txx} + \gamma u^b u_x + \sigma u^{b-1} u_x u_{xx} + \delta u^b u_{xxx})] = 0$$

if and only if $\sigma = (b+1)\delta$ or $b = 0$.

Substituting $\sigma = (b+1)\delta$ into (21), after reckoning one obtains

$$u_t + \varepsilon u_{txx} + \gamma u^b u_x + (b+1)\delta u^{b-1} u_x u_{xx} + \delta u^b u_{xxx}$$

$$= D_t\left(u^2 - \varepsilon u_x^2\right) + D_x\left(\frac{2}{2+b}u^{b+2} - 2\delta u^{b+1}u_{xx} + 2\varepsilon u u_{tx}\right),$$

if $b \neq -2$. This means that the vector field

$$C^0 = u^2 - \varepsilon u_x^2, \quad C^1 = \frac{2}{2+b}u^{b+2} - 2\delta u^{b+1}u_{xx} + 2\varepsilon u u_{tx} \tag{25}$$

is a conserved vector of equation

$$u_t + \varepsilon u_{txx} + \gamma u^b u_x + \delta(b+1)u^{b-1}u_x u_{xx} + \delta u^b u_{xxx} = 0, \tag{26}$$

when $b \neq -2$. Then, for this case, the theorem follows from Remark 1.

Whenever $b = -2$, a straightforward calculation shows that the components (25) become

$$C^0 = u^2 - \varepsilon u_x^2, \quad C^1 = \gamma \ln u - 2\delta\frac{u_{xx}}{u} + 2\varepsilon u u_{tx} \tag{27}$$

and this proves the theorem. □

Equation (26) includes some important equations. In the next table we present some members of it as well as its corresponding components of the conserved vector (25).

ε	b	γ	δ	Equation	Conserved density	Conserved flux
-1	1	-1	0	BBM	$u^2 + u_x^2$	$\frac{2}{3}u^3 - 2uu_{tx}$
-1	1	3	1	Camassa-Holm	$u^2 + u_x^2$	$2u^3 - 2u^2u_{xx} - 2uu_{tx}$
-1	2	4	1	Novikov	$u^2 + u_x^2$	$2u^4 - 2u^3u_{xx} - 2uu_{tx}$
0	$\neq -2$	$\forall$	0	Riemman	u^2	$\frac{2}{2+b}\gamma u^{b+2}$

TABLE 1. In this table it is presented some equations of the type (26) as well as some conserved currents.

Choosing $\varepsilon = -1$, $\gamma = \beta(b+2)$ and $t \mapsto \beta^{-1}t$, equation (26) can be rewritten as

$$u_t - u_{txx} + (b+2)u^b u_x = (b+1)u^{b-1}u_x u_{xx} + u^b u_{xxx}. \tag{28}$$

Equation (28) is a one-parameter family of equations connecting Camassa-Holm (3) and Novikov (6) equations, which correspond, respectively, to the cases $b = 1$ and $b = 2$.

6. **Conclusions.**

In this paper we deduced the equation (28), which provides a one-parameter family of equations unifying both Camassa-Holm and Novikov equations. It is known that such equations are completely integrable [18], which means that both admit a bi-hamiltonian structure.

Although the family (28) is, probably, the first one-parameter equation connecting CH and Novikov equations, it is not *the first* one-parameter family of equations connecting two integrable equations, since there is the well known $b-$equation. Here we would like to compare (28) and the $B-$equation[1] (see [15, 17])

$$u_t - u_{txx} + (B+1)uu_x = Bu_x u_{xx} + uu_{xxx}. \tag{29}$$

Clearly (29) admits the scale invariance $(x, t, y) \mapsto (x, \lambda^{-1}t, \lambda u)$, since it can be obtained from the family (21) choosing $b = 1$, $\gamma = B + 1$, $\sigma = -B$ and $\delta = 1$. However, comparing

[1]In fact, in the references the equation is denoted by $u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}$. However, here, in order to avoid confusion, we use the form (29).

(29) with (28), we conclude that $b = 1$ and $B = 2$, which means that (29), with these values, is the Camassa-Holm equation.

In [15] it was shown that (29) is also integrable if $B = 3$. However, such equation is not a member of our family (28). In fact, comparing (29) with (11) it is easy to conclude that $f(u) = (B + 1)u$, $g(u) = -B$ and $h(u) = -u$, but it can only be written in a conservation law after multiplied by u when $B = 2$, which is the CH equation.

Although (29) possesses among its members, CH and Degasperis-Procesi equations (case $B = 3$), which are both integrable, it only admits the multiplier u in the condition (17) for a specific B, actually, the corresponding one to the CH equation. Moreover, these are the only integrable equations of the type (29), see [15]. On the other hand, our new equation (28) also connects at least two integrable equations, namely, Camassa-Holm and Novikov. But, differently from (29), every member of (28) admits the multiplier $\mu = u$.

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