ISSN 2317-3297

Some results on Lie symmetry analysis of a fourth-order Emden-Fowler equation

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Key-words: *Emden-Fowler equation, Lie symmetries, exact solutions*

Abstract: *Some recent results on Lie group analysis of a class of fourth-order ordinary differential equations are revisited. A complete group classification is carried out. Then some group invariant solutions of the equation under consideration are investigated. Next it is shown how to construct exact solutions to a class of Lane-Emden systems from the group-invariant solutions of the fourth-order Emden-Fowler equations considered.*

1 Introduction

The purpose of this paper is to revisit some recent results on Lie symmetry analysis [2] obtained by the authors with respect to the class of equations

$$y'''' + ax^{\gamma}y^{p} = 0. \tag{1}$$

In (1) $x \in (-\infty, +\infty)$ and $y = y(x)$ denote, respectively, the independent and dependent variables while a, γ and p are real constitutive parameters.

These equations fall in the wider class of the so-called *fourth-order Emden-Fowler equation* and they appear in several problems in Mathematics, Physics and Engineering.

For instance the relationship between the applied load and the deflection in a non-homogeneous beam is described by an equation of the type (1).

On the other hand, equations of the type (1) arise, as solving equations, from stationary weakly coupled reaction-diffusion systems. These equations have also been studied by several authors which have focused their attention in properties of its solutions. For further details, see [2] and references therein.

2 Group classification

Let I be an interval on the line R and $y : I \to R$ be a smooth function. A Lie point symmetry of a given ordinary differential equation

$$F(x, y, \cdots, y^{(n)}) = 0, \tag{2}$$

ISSN 2317-3297

with

$$y^{(k)} := \frac{d^k y(x)}{dx^k}, \quad 1 \le k \le n, \quad y^{(0)} := y,$$

is a local group of diffeomorfisms on $I \times R$ which preservs the equation. The set of all Lie point symmetries of (2) generates the *Lie point symmetry group* G of equation (2).

For any Lie point symmetry g_X of equation (2) there exists a corresponding Lie point symmetry generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{3}$$

If X is a Lie point symmetry generator of (2), it is said that X is admitted by (2). A necessary and sufficient condition for (3) to be admitted by (2) is

$$X^{(n)}F = \lambda F, \tag{4}$$

for a certain function $\lambda = \lambda(x, y, y', \cdots)$, where

$$X^{(n)} := X + \sum_{j=1}^{n} \eta^{(j)} \frac{\partial}{\partial y^{(j)}}, \tag{5}$$

$$\eta^{(j)} := D\eta^{(j-1)} - y^{(j)} D\xi, \;\; j \ge 1 \tag{6}$$

and

$$D := \frac{\partial}{\partial x} + \sum_{j=1}^{\infty} y^{(j)} \frac{\partial}{\partial y^{(j-1)}}. \tag{7}$$

The operator D is called total derivative operator and equation (4) is the *invariance condition.*

The main result obtained in [2] can be summarized by the following table.

	p	γ	a	Generators
$N1$	$\forall$	$\neq 0, -(5+3p)$	$\neq 0$	$(1-p)x\partial_x + (4+\gamma)y\partial_y$
$N2$	$p \neq -5/3$	$-(5+3p)$	$\neq 0$	$(p-1)x\partial_x + (3p+1)y\partial_y,\; x^2\partial_x + 3xy\partial_y$
$N3$	$\forall$	0	$\neq 0$	$\partial_x,\; (1-p)x\partial_x + 4y\partial_y$
$N4$	$-5/3$	$\neq 0$	$\neq 0$	$x\partial_x + \frac{12+3\gamma}{8}y\partial_y$
$N5$	$-5/3$	0	$\neq 0$	$\partial_x,\; x\partial_x + \frac{3}{2}y\partial_y,\; x^2\partial_x + 3xy\partial_y$
$L1$	$-$	$-$	0	$\partial_x,\, \partial_y,\, x\partial_x,\, y\partial_y,\, x\partial_y,\, x^2\partial_x,\, x^3\partial_y,\, x^2\partial_x + 3xy\partial_y$
$L2$	0	$\forall$	$\neq 0$	*this case is equivalent to the case* $a = 0$ *under the change* $y \mapsto y - \frac{ax^{\gamma+4}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}$
$L3$	1	0	1	$\partial_x,\, y\partial_y,\, e^x\partial_y$ $e^{-x}\partial_y,\, \sin x\partial_y,\, \cos x\partial_y$
$L4$	1	0	-1	$\partial_x,\, y\partial_y,\, e^{\frac{x}{\sqrt{2}}} \sin \frac{x}{\sqrt{2}}\partial_y,$ $e^{\frac{x}{\sqrt{x}}} \cos \frac{x}{\sqrt{2}}\partial_y,\; e^{-\frac{x}{\sqrt{x}}} \cos \frac{x}{\sqrt{2}}\partial_y,\; e^{\frac{-x}{\sqrt{2}}} \sin \frac{x}{\sqrt{2}}\partial_y$
$L5$	1	$\neq -4, -8$	a	$\beta(x)\partial_y$
$L6$	1	-4	a	$x\partial_x + \frac{3}{2}y\partial_y,\; \beta(x)\partial_y$
$L7$	1	-8	a	$x^2\partial_x + 3xy\partial_y,\; \beta(x)\partial_y$

Tabela 1: Group classification of equation (1).

Remark: In the cases $L5$, $L6$ and $L7$, the function $\beta(x)$ satisfies the equation

$$\beta'''' + ax^{\gamma}\beta = 0,$$

whose solutions can be found in terms of the Mittag-Leffler functions under an appropriate transformation.

ISSN 2317-3297

3 Invariant solutions of equations of the class (1)

In this section we show some nontrivial invariant solutions of equation (1), with $p \neq 0, 1$, obtained from the results of the previous sections.

First consider the case $\gamma = 0$ and $a \neq 0$. Then (1) becomes

$$y'''' + ay^p = 0.$$

The Lie point symmetry generator

$$X = (1-p)x\frac{\partial}{\partial x} + 4y\frac{\partial}{\partial y}$$

allows us to get the following solution:

$$y(x) = \left[-\frac{8}{a}\frac{(p+3)(p+1)(3p+1)}{(1-p)^4}\right]^{\frac{1}{p-1}} x^{\frac{4}{1-p}}.$$

Now consider the equation

$$y'''' + ax^{-(5+3p)}y^p = 0.$$

From the Lie point symmetry generator

$$X = (p-1)x\frac{\partial}{\partial x} + (3p+1)y\frac{\partial}{\partial y}$$

we obtain the invariant solution

$$y(x) = \left[-\frac{8}{a}\frac{(p+3)(p+1)(3p+1)}{(p-1)^4}\right]^{\frac{1}{p-1}} x^{\frac{3p+1}{p-1}}.$$

Concerning the equation

$$y'''' + ay^{-\frac{5}{3}} = 0$$

and the Lie point symmetry generator

$$X = x\frac{\partial}{\partial x} + \frac{3}{2}y\frac{\partial}{\partial y}$$

we find the solution

$$y(x) = \left(-\frac{16a}{9}\right)^{3/8} x^{\frac{3}{2}}.$$

For $\gamma \neq 0$ and $p = -5/3$ the equation (1) becomes

$$y'''' + ax^{\gamma}y^{-\frac{5}{3}} = 0 \tag{8}$$

and the generator

$$X = x\frac{\partial}{\partial x} + \frac{12+3\gamma}{8}y\frac{\partial}{\partial y}$$

gives the solution

$$y(x) = \left[-\frac{4096a}{(3\gamma+12)(3\gamma+4)(3\gamma-4)(3\gamma-12)}\right]^{3/8} x^{\frac{3\gamma+12}{8}},$$

provided that $(3\gamma+12)(3\gamma+4)(3\gamma-4)(3\gamma-12) \neq 0$.

Finally we consider the equation

$$y'''' + ax^{\gamma}y^p = 0,$$

with p arbitrary and $\gamma \neq 0, -(5+3p)$. From the generator

$$X = (1-p)x\frac{\partial}{\partial x} + (4+\gamma)y\frac{\partial}{\partial y}$$

we obtain the solution

$$y(x) = \left[-\frac{(4+\gamma)(3+\gamma+p)(2+\gamma+2p)(1+\gamma+3p)}{a(1-p)^4}\right]^{\frac{1}{p-1}} x^{\frac{4+\gamma}{1-p}}.$$

ISSN 2317-3297

4 Exact solutions to a class of Lane-Emden systems

It is a straightforward calculation to see that the class of Lane-Emden systems

$$\begin{cases} u''(x) + v(x) = 0, \\ \\ v''(x) + x^{\gamma} u(x)^{p} = 0 \end{cases} \tag{9}$$

is equivalent to equation (1) with $a = -1$. For furhter details on Lane-Emden systems, see [1] and references therein.

Here we use the group invariant solutions of the equation (1) to construct exact solutions to the Lane-Emden system

$$\begin{cases} u''(x) + v(x) = 0, \\ \\ v''(x) + x^{\gamma} u(x)^{p} = 0. \end{cases} \tag{10}$$

From the first equation of (10) easily follows

$$v = -u'' \tag{11}$$

so from second equation of (10) we get

$$u''' - x^{\gamma} u^{p} = 0. \tag{12}$$

If we take $p = -5/3$ in (12) we obtain a particular case of the equation (8), whose invariant solution is

$$u_{\gamma}(x) = \left[\frac{4096}{(3\gamma+12)(3\gamma+4)(3\gamma-4)(3\gamma-12)}\right]^{3/8} x^{\frac{3\gamma+12}{8}}. \tag{13}$$

From (11) and (13) we obtain

$$v_{\gamma} = -\frac{[(3\gamma+12)(3\gamma+4)]^{5/8}}{[(3\gamma-12)(3\gamma-4)]^{3/8}} x^{\frac{3\gamma-4}{8}}. \tag{14}$$

Thus (13) and (14) provide a family of solutions to the systems (10).

In particular

$$u(x) = \left(\frac{4}{3}\right)^{3/4} x^{3/2}, \quad v(x) = \left(\frac{3}{4}\right)^{1/4} \frac{1}{\sqrt{x}}$$

are solutions of the system

$$\begin{cases} u''(x) + v(x) = 0, \\ \\ v''(x) + u(x)^{-\frac{5}{3}} = 0. \end{cases} \tag{15}$$

Acknowledgements

The authors would like to thank FAPESP for financial support (grants 2010/10259-3, 2011/20072-0 and 2011/19089-6). Mariano Torrisi would also like to thank CMCC-UFABC for its warm hospitality and GNFM (*Gruppo Nazionale per Fisica-Matematica*) for its support.

Referências

[1] Y. Bozhkov and I. L. Freire, On the Lane-Emden system in dimension one, Appl. Math. Comp., 218, (2012), 10762–10766, DOI: 10.1016/j.amc.2012.04.033.

[2] I. L. Freire, P. L. Silva and M. Torrisi, Lie and Noether symmetries for a class of fourth-order Emden-Fowler equations, (2012), submitted.