Lie and Noether symmetries for a class of fourth-order Emden-Fowler equations

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Abstract

A group classification of a fourth-order ordinary differential equation is carried out. The Noether symmetries are considered and some first integrals are established. Solutions for special Lane-Emden systems are also obtained from the invariant solutions of the investigated equation.

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1 Introduction

The aim of this paper is to study the class of equations

$$y'''' + ax^{\gamma}y^{p} = 0 \tag{1}$$

from the point of view of symmetries. In (1), $x \in \mathbb{R}$ and $y = y(x)$ denote, respectively, the independent and dependent variables while a, γ and p are real constitutive parameters.

These equations are called *fourth-order Emden-Fowler equations* and they appear in several problems in Mathematics, Physics and Engineering, see [1, 2, 3, 4]. An equation of the type (1), for instance, describes the relationship between the applied load and the deflection in a non-homogeneous beam. For further details and related phenomena, see [1].

Such as the second-order Emden-Fowler equation $y'' = f(x)y^{n}$, it is possible to verify that (1) can be obtained from a Lagrangian. In this case, such Lagrangian takes the form:

$$\mathcal{L} = \frac{(y'')^2}{2} + ax^{\gamma}F(y), \quad F(y) = \begin{cases} \dfrac{y^{p+1}}{p+1}, & p \neq -1, \\ \\ \ln|y|, & p = -1. \end{cases} \tag{2}$$

Inspired by some works, see [5, 6, 7], concerned with the second-order Emden-Fowler equations and taking into account the analysis about $y'''' = f(y)$ performed in [2], in this paper we consider (1) in the framework of Lie and Noether symmetries.

In Section 2 we carry out a complete group classification of (1). In Section 3 we classify those Lie point symmetries in the context of Noether symmetries and obtain their corresponding first integrals. Invariant solutions for equation (1) are obtained in Section 4. Furthermore, from these exact solutions, and taking into account that the Lane-Emden system

$$\begin{cases} u''(x) + v(x) = 0, \\ \\ v''(x) + x^{\gamma}u(x)^{p} = 0 \end{cases} \tag{3}$$

can be reduced to an equation of the type (1), we construct some exact solutions for (3).

Lane-Emden systems with one dependent variable were considered in [8, 9]. However, in those references no solutions for systems of the type (3) were provided. This is, to the best of our knowledge, the first time that invariance properties of a scalar equation are employed in order to find solutions to systems of the type (3).

Conclusions are given in section 5.

2 Lie point symmetries

Here we carry out a complete group classification of equation (1). In the following we provide some short elements about the Lie symmetry application to ODEs, however for further details we address the interested reader to the well known monographs [10, 11, 12, 13, 14].

Let $I \subseteq \mathbb{R}$ an interval and $y = y(x)$, $x \in I$, be a real smooth function. A Lie point symmetry of a given ordinary differential equation

$$F(x, y, \cdots, y^{(n)}) = 0, \tag{4}$$

with

$$y^{(k)} := \frac{d^k y(x)}{dx^k}, \quad 1 \le k \le n, \quad y^{(0)} := y,$$

is a local continuous invertible transformation on $I \times \mathbb{R}$

$$\hat{x} = \hat{x}(x, y, \varepsilon) \qquad \hat{y} = \hat{y}(x, y, \varepsilon),$$

characterized by the value of the parameter ε, that leaves invariant the solutions of the considered equation. If the parameter varies continuously, the set of all Lie point symmetries of (4) described by a real parameter ε generates the *Lie point continuous symmetry group* G of equation (4).

For any Lie point symmetry of equation (4) there exists a corresponding Lie point symmetry generator

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y, \tag{5}$$

where

$$\xi(x, y) = \left.\frac{\partial \hat{x}(x, y, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \eta(x, y) = \left.\frac{\partial \hat{y}(x, y, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

If X is a Lie point symmetry generator of (4), it is said that X is admitted by (4). A necessary and sufficient condition for (5) to be admitted by (4) is

$$X^{(n)} F = \lambda F, \tag{6}$$

for a certain function $\lambda = \lambda(x, y, y', \cdots)$, where

$$X^{(n)} := X + \sum_{j=1}^{n} \eta^{(j)} \frac{\partial}{\partial y^{(j)}}, \tag{7}$$

$$\eta^{(j)} := D\eta^{(j-1)} - y^{(j)} D\xi, \;\; j \ge 1 \tag{8}$$

and

$$D := \frac{\partial}{\partial x} + \sum_{j=1}^{\infty} y^{(j)} \frac{\partial}{\partial y^{(j-1)}}. \tag{9}$$

The operator D is called total derivative operator and equation (6) is the *invariance condition*.

Considering the structure of equations (1) and according to Theorem 3.3.4-2 of [11] (see also [15]), we observe that any Lie point symmetry generator admitted by (1) takes the form

$$X = \xi(x)\partial_x + [\alpha(x)y + \beta(x)]\partial_y. \tag{10}$$

After calculating the extensions $\eta^{(1)}$, $\eta^{(2)}$, $\eta^{(3)}$, $\eta^{(4)}$ in order to obtain the fourth-extension $X^{(4)}$ of the generator (10) and considering $F = y'''' + ax^\gamma y^p$, from the invariance condition (6), we obtain

$$\beta''''(x) + pax^\gamma \beta(x) y^{p-1} + [pax^\gamma \alpha(x) + a\gamma x^{\gamma-1}\xi(x)]y^p + \alpha''''(x)y + [4\alpha'''(x) - \xi''''(x)]y'$$

$$+[6\alpha''(x) - 4\xi'''(x)]y'' + [4\alpha'(x) - 6\xi''(x)]y''' + [\alpha(x) - 4\xi'(x)]y'''' = \lambda[y'''' + ax^\gamma y^p].$$

Thus, the *determining system* reads

$$4\alpha'''(x) - \xi''''(x) = 0, \quad 6\alpha''(x) - 4\xi'''(x) = 0, \quad 4\alpha'(x) - 6\xi''(x) = 0, \tag{11}$$

$$\alpha(x) - 4\xi'(x) = \lambda, \tag{12}$$

$$\beta''''(x) + pax^\gamma \beta(x) y^{p-1} + [pax^\gamma \alpha(x) + a\gamma x^{\gamma-1}\xi(x)]y^p + \alpha''''(x)y = \lambda a x^\gamma y^p. \tag{13}$$

Solving equations (11), we get

$$\xi(x) = c_0 + c_1 x + c_2 x^2, \quad \alpha(x) = 3c_2 x + \frac{3}{2}c_1 + k,$$

where c_0, c_1, c_2 and k are arbitrary constants.

From the invariance condition, any Lie point symmetry generator admitted by (1) is written as

$$X = (c_0 + c_1 x + c_2 x^2)\partial_x + [(3c_2 x + \frac{3}{2}c_1 + k)y + \beta(x)]\partial_y,$$

where c_0, c_1, c_2 and k are constants satisfying, taking (12) and (13) into account, the following *classifying equation* is obtained

$$\begin{aligned} &\beta''''(x) + pax^\gamma \beta(x) y^{p-1} + \\ &\left\{ a\left[\frac{5+3p}{2}c_1 + k(p-1) + (5+3p)c_2 x\right] x^\gamma + a\gamma(c_0 + c_1 x + c_2 x^2)x^{\gamma-1} \right\} y^p = 0 \end{aligned} \tag{14}$$

for the constitutive parameters p, γ, a and the unknown function β.

We observe that from (14) we have many possibilities for analyzing the parameters p, γ and a. Therefore, from this analysis it is obtained the complete classification of (1), which is summarized on Table 1.

	p	γ	a	Generators
$N1$	$\forall$	$\neq 0, -(5+3p)$	$\neq 0$	$(1-p)x\partial_x + (4+\gamma)y\partial_y$
$N2$	$p \neq -5/3$	$-(5+3p)$	$\neq 0$	$(p-1)x\partial_x + (3p+1)y\partial_y,\ x^2\partial_x + 3xy\partial_y$
$N3$	$\forall$	0	$\neq 0$	$\partial_x,\ (1-p)x\partial_x + 4y\partial_y$
$N4$	$-5/3$	$\neq 0$	$\neq 0$	$x\partial_x + \frac{12+3\gamma}{8}y\partial_y$
$N5$	$-5/3$	0	$\neq 0$	$\partial_x,\ x\partial_x + \frac{3}{2}y\partial_y,\ x^2\partial_x + 3xy\partial_y$
$L1$	$-$	$-$	0	∂_x, ∂_y, $x\partial_x$, $y\partial_y$, $x\partial_y$, $x^2\partial_y$, $x^3\partial_y$, $x^2\partial_x + 3xy\partial_y$
$L2$	0	$\forall$	$\neq 0$	this case is equivalent to the case $a = 0$ under the change $y \mapsto y - \frac{ax^{\gamma+4}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}$
$L3$	1	0	1	∂_x, $y\partial_y$, $e^{\frac{x}{\sqrt{2}}}\sin\frac{x}{\sqrt{2}}\partial_y,\ e^{\frac{x}{\sqrt{2}}}\cos\frac{x}{\sqrt{2}}\partial_y,\ e^{-\frac{x}{\sqrt{2}}}\cos\frac{x}{\sqrt{2}}\partial_y,\ e^{\frac{-x}{\sqrt{2}}}\sin\frac{x}{\sqrt{2}}\partial_y$
$L4$	1	0	-1	∂_x, $y\partial_y$, $e^x\partial_y$, $e^{-x}\partial_y$, $\sin x\partial_y$, $\cos x\partial_y$
$L5$	1	$\neq -4, -8$	$\neq 0$	$y\partial_y$, $\beta(x)\partial_y$
$L6$	1	-4	$\neq 0$	$y\partial_y,\ x\partial_x + \frac{3}{2}y\partial_y,\ \beta(x)\partial_y$
$L7$	1	-8	$\neq 0$	$y\partial_y$, $x^2\partial_x + 3xy\partial_y,\ \beta(x)\partial_y$

Table 1: Group classification of equation (1). Some of the linear cases (L1, L3, L4) were obtained in [2]. We only recall them here for sake of completeness.

Remark: In the cases $L5$, $L6$ and $L7$, the function $\beta(x)$ satisfies the equation $\beta'''' + ax^\gamma\beta = 0$, whose solutions can be found in terms of the Mittag-Leffler functions.

3 Noether symmetries and first integrals

As we have already mentioned in the Introduction, equation (1) is the Euler-Lagrange equation

$$\frac{\delta \mathcal{L}}{\delta y} = 0,$$

where $\mathcal{L}$ is given by (2), while

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} - D\frac{\partial}{\partial y'} + D^2\frac{\partial}{\partial y''}$$

is the Euler-Lagrange operator and D is the total derivative operator, given by (9).

Whenever an equation possesses variational structure, *i.e.*, it is an Euler-Lagrange equation, we can also investigate a special class of symmetries which allows us to construct some invariants (first integrals) preserved on the solutions of the equation. Such class of symmetries is called Noether symmetries.

A point symmetry operator $X = \xi\partial_x + \eta\partial_y$ is a Noether symmetry operator of equation (1) if the identity

$$X^{(2)}\mathcal{L} + \mathcal{L}D\xi = DA \tag{15}$$

holds for a certain function $A = A(x, y, y')$, called *potential*, where the Lagrangian $\mathcal{L}$ is given by (2).

Whenever A is a non-constant function, the operator X is called *divergence symmetry*. If $A =$ *const.*, the operator is called *variational symmetry*.

Finally, if X is a Noether point symmetry generator, following the celebrated Noether theorem [16], we can affirm that the corresponding *first integral* of (1), associated to X, is given by

$$I = A - \xi\mathcal{L} - (\eta - y'\xi)\frac{\delta}{\delta y'}\mathcal{L} - D(\eta - y'\xi)\frac{\delta}{\delta y''}\mathcal{L}, \tag{16}$$

where $\mathcal{L}$ is the Lagrangian (2) of the family of equations (1) and $\delta/\delta y'$, $\delta/\delta y''$ are the Euler-Lagrange operators with respect to y' and y'', respectively.

If (16) is a first integral of the equation (1), it means that the quantity I given by (16) is conserved on the solutions $y = y(x)$ of (1), which means

$$DI|_{y''''+ax^\gamma y^p=0} \equiv 0. \tag{17}$$

In particular, it follows that the change $A \mapsto A + k$ in (16), where k is any constant, leaves (17) invariant. For this reason, for variational symmetries we can always take $A = 0$ in (15).

Remark 1: We would like to recall that we obtain here the Noether symmetry generators from the Lie point symmetry generators. In general these sets, Lie and Noether symmetries, are different. In fact, just in the following, the reader can find equations for which the set of Noether symmetries does not coincide with the set of Lie point symmetries.

Remark 2: Lie point symmetries allow us to find invariants which are solutions of the considered equation. Noether symmetry, according to (16), gives a first integral, *i.e*, a differential equation whose order is lower than the original one. The solutions of this last equation are also solutions of the original one, see, *e.g.*, [2].

Here we consider the Lie point symmetries found in the nonlinear cases (see Table 1) of equations (1) in order to ascertain which of them are Noether symmetries. The interested reader can find the linear cases of (1) in [2].

After having applied formula (15) we can affirm:

Theorem 1. *The following statements hold.*

1. The translational symmetry in x is a variational symmetry of the equation

$$y'''' + ay^p = 0, \tag{18}$$

for any p.

2. The Lie point symmetry generator for the equation (18)

$$X_p = (1-p)x\partial_x + 4y\partial_y \tag{19}$$

is a variational symmetry if and only if $p = -5/3$.

3. The Lie point symmetry generator

$$X = x^2\partial_x + 3xy\partial_y \tag{20}$$

is a divergence symmetry of the equation

$$y'''' + ax^{-(3p+5)}y^p = 0. \tag{21}$$

In particular, if $p = -5/3$, then X is a divergence symmetry of the equation $y'''' + ay^{-\frac{5}{3}} = 0$.

Proof. The first and the third cases are mere straightforward calculations and we leave the details to the interested reader. With respect to the second case, let $X^{(2)}$ be the second extension of (19). Then it is easy to check that

$$X_p^{(2)}\mathcal{L} + \mathcal{L}D\xi = (3p+5)\mathcal{L}. \tag{22}$$

Assume that X_p is a variational symmetry. Then the first member of (22) must be 0 and, as a consequence, $p = -5/3$.

On the other hand, if $p = -5/3$, the second side of (22) vanishes. Then X_p is a variational symmetry. □

Therefore, by summarizing, in the nonlinear cases we can affirm that:

1.) The translation in x is a Noether symmetry in the cases N_3 and N_5.

2.) Even though there is more than one case admitting the scale symmetry (19), only in the case N_5 it is also a Noether symmetry.

3.) The symmetry (20) is a Noether symmetry for any equation of the type (1) admitting such an operator.

4.) As a consequence of the previous observations, all Lie point symmetries of the equation $y'''' + ay^{-\frac{5}{3}} = 0$ are Noether symmetries.

By using (16), we obtain the first integrals associated with the Noether symmetries of the nonlinear cases of equation (1). In the Table 2 we present the Noether symmetries obtained as well as their corresponding potentials and first integrals.

Equation	Generator X	Potential A	First Integrals I
$y'''' + ay^{-\frac{5}{3}} = 0$	∂_x	0	$\frac{(y'')^2}{2} + \frac{3}{2}ay^{-2/3} - y'y'''$
$y'''' + ay^{-\frac{5}{3}} = 0$	$x\partial_x + \frac{3}{2}y\partial_y$	0	$\frac{x(y'')^2}{2} + \frac{3}{2}axy^{-2/3} - xy'y''' + \frac{3}{2}yy''' + \frac{1}{2}y'y''$
$y'''' + ay^{-\frac{5}{3}} = 0$	$x^2\partial_x + 3xy\partial_y$	$2(y')^2$	$\frac{x^2(y'')^2}{2} + \frac{3}{2}ax^2y^{-2/3} - x^2y'y'''$ $+3xyy''' - xy'y'' - 3yy'' + 2(y')^2$
$y'''' + ax^{-(3p+5)}y^p = 0$	$x^2\partial_x + 3xy\partial_y$	$2(y')^2$	$2(y')^2 + \frac{x^2(y'')^2}{2} - ax^{-3(p+1)}\frac{y^{p+1}}{p+1}$ $+(3xy - y'x^2)y''' - (3y + xy')y''$
$y'''' + ax^{-2}y^{-1} = 0$	$x^2\partial_x + 3xy\partial_y$	$2(y')^2 + 3a\ln\|x\|$	$2(y')^2 + 3a\ln\|x\| - a\ln\|y\| + \frac{x^2(y'')^2}{2}$ $+(3xy - y'x^2)y''' - (3y + xy')y''$

Table 2: Lie point symmetries of equation (1), with $a \neq 0$, $p \neq 0, 1$, which are Noether symmetries. At the penultimate row, $p \neq -1$. The cases corresponding to $p = -5/3$ can be found in [2]. The last two first integrals are new.

4 Exact Solutions

In this section we use the method of characteristics [10] to find some nontrivial invariant solutions of equation (1), with $p \neq 0, 1$.

For constructing solutions, we will proceed as the follows:

1. To begin with, we construct an invariant on the $x - y$ plane. To do this, let

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

   be a Lie point symmetry generator. We say that $y = f(x)$ is invariant under X if

$$X(y - f(x))|_{y=f(x)} \equiv 0. \tag{23}$$

2. Condition (23) is equivalent to find the solution of the following ordinary differential equation

$$\eta - \xi y' = 0, \tag{24}$$

   called *characteristic equation.*

3. The general solution of (24) is, therefore, substituted in (1). Then, by requiring that the general solution of (24) satisfies (1) identically, we are able to select the invariant solutions of (1).

4.1 Invariant solutions of equations (1)

Consider the equation (18) and the generator (19). Substituting the solution of the corresponding characteristic equation into (18) we get the following solution

$$y(x) = \left[-\frac{8}{a}\frac{(1+p)(3+2p)(1+3p)}{(p-1)^4}\right]^{\frac{1}{p-1}} x^{\frac{4}{1-p}}.$$

Regarding the equation $y'''' + ay^{-\frac{5}{3}} = 0$, we consider the linear combination $X = \lambda X_1 + \alpha X_2 + \delta X_3$ of the generators

$$X_1 = \partial_x, \; X_2 = x\partial_x + \frac{3}{2}y\partial_y, \; X_3 = x^2\partial_x + 3xy\partial_y,$$

where λ, α, δ are arbitrary parameters.

For this new generator and using the same procedure employed for the previous solution, the following three-parameter family of solutions is obtained:

$$y(x) = \left[\frac{-16a}{9(\alpha^2 - 4\lambda\delta)^2}\right]^{3/8} (\lambda + \alpha x + \delta x^2)^{3/2}, \tag{25}$$

with $\alpha^2 \neq 4\lambda\delta$.

For $\gamma \neq 0$ and $p = -5/3$, the equation (1) becomes

$$y'''' + ax^\gamma y^{-\frac{5}{3}} = 0 \tag{26}$$

and the generator $X = x\partial_x + \frac{12+3\gamma}{8}y\partial_y$ gives the solution

$$y(x) = \left[-\frac{4096a}{(3\gamma + 12)(3\gamma + 4)(3\gamma - 4)(3\gamma - 12)}\right]^{3/8} x^{\frac{3\gamma+12}{8}},$$

with $(3\gamma + 12)(3\gamma + 4)(3\gamma - 4)(3\gamma - 12) \neq 0$.

Finally we consider the equation $y'''' + ax^\gamma y^p = 0$, for any $p \neq 1$ and $\gamma \neq 0, -(5 + 3p)$. From the generator $X = (1 - p)x\partial_x + (4 + \gamma)y\partial_y$ we obtain the solution

$$y(x) = \left[-\frac{(4 + \gamma)(3 + \gamma + p)(2 + \gamma + 2p)(1 + \gamma + 3p)}{a(1 - p)^4}\right]^{\frac{1}{p-1}} x^{\frac{4+\gamma}{1-p}}.$$

4.2 Solutions to the Lane-Emden systems

Here we use the group invariant solutions of the equation (1) to construct exact solutions to the Lane-Emden system (3). According to the results of the previous section,

$$u(x) = \left[\frac{4096}{(3\gamma + 12)(3\gamma + 4)(3\gamma - 4)(3\gamma - 12)}\right]^{3/8} x^{\frac{3\gamma+12}{8}} \tag{27}$$

provides a family of solutions to (26). Therefore, invoking the first equation of (3), we obtain

$$v = -2^{-3/2}\frac{[(3\gamma + 12)(3\gamma + 4)]^{5/8}}{[(3\gamma - 12)(3\gamma - 4)]^{3/8}} x^{\frac{3\gamma-4}{8}}. \tag{28}$$

Thus (27) and (28) give a family of solutions to the system (3) with $p = -5/3$ and $\gamma \neq 0$.

Invoking (25), we can easily obtain the three-parameter family of solutions

$$\begin{aligned} u(x) &= \left[\frac{-16a}{9(\alpha^2 - 4\lambda\delta)^2}\right]^{3/8} (\lambda + \alpha x + \delta x^2)^{3/2}, \\ v(x) &= -\frac{3}{4}\left[\frac{-16a}{9(\alpha^2 - 4\lambda\delta)^2}\right]^{3/8} (\lambda + \alpha x + \delta x^2)^{-1/2}(\alpha^2 + 4\delta\lambda + 8\alpha\delta x + 8\delta^2 x^2) \end{aligned} \tag{29}$$

to the system

$$\begin{cases} u''(x) + v(x) = 0, \\ \\ v''(x) + u(x)^{-\frac{5}{3}} = 0. \end{cases} \tag{30}$$

Lane-Emden systems of the type (30) with power nonlinearities in both equations were studied in [8, 9] from the point of view of Lie and Noether symmetries and first integrals. In [9] some solutions were constructed. However, in both references there are no solutions to Lane-Emden systems of the type (30), that is, with one linear equation. Furthermore, following the procedure used to obtain the solutions (27) – (29), it is possible to construct more solutions for some systems studied in [8, 9].

From the previous results we now construct a wide family of invariant solutions of a class of bidimensional Lane-Emden systems

$$\begin{cases} u_{xx} + u_{yy} + v^p = 0, \\ \\ v_{xx} + v_{yy} + u^q = 0. \end{cases} \tag{31}$$

We observe that choosing $p = 1$ and $q = -5/3$ and looking for solution of the type $u = u(x)$ and $v = v(x)$, (31) is reduced to the system (30). Some solutions for (30) are given by (29). Since the system (31) is invariant under rotations on the plane $x - y$, we can easily construct solutions of (31) depending on x and y by applying the transformation

$$\begin{aligned} u(x, y, \varepsilon) &= u(x\cos\varepsilon - y\sin\varepsilon), \\ \\ v(x, y, \varepsilon) &= u(x\cos\varepsilon - y\sin\varepsilon), \end{aligned} \tag{32}$$

on (29).

5 Conclusions

In this paper we have considered a class of fourth-order ordinary differential equations, describing several phenomena, from the point of view of symmetries. Moreover we would like to observe that we have ascertained which Lie point symmetries, admitted from the nonlinear cases of the class of equations (1), are Noether symmetries. In fact, there are even several papers on the Noether classification for scalar second-order ODEs, the literature on Noether symmetries of fourth-order ODEs is scarce [2, 3]. First integrals for equation (1) have been obtained and several classes of exact solutions are exhibited.

We would like to point out that the investigated equation is employed for modelling the applied load and the deflection in a non-homogenous beam when the force on the beam depend on the deflection and it may also be dependent on the position. In this case the solutions obtained in the section 4 describe the behaviour of the deflection at a point x.

Finally we have showed how to construct, from the solutions of the equation (1), solutions to the one and bidimensional Lane-Emden systems.

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