

A FAMILY OF WAVE-BREAKING EQUATIONS GENERALIZING THE CAMASSA-HOLM AND NOVIKOV EQUATIONS

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ABSTRACT. A 4-parameter polynomial family of equations generalizing the Camassa-Holm and Novikov equations that describe breaking waves is introduced. A classification of low-order conservation laws, peaked travelling wave solutions, and Lie symmetries is presented for this family. These classifications pick out a 1-parameter equation that has several interesting features: it reduces to the Camassa-Holm and Novikov equations when the polynomial has degree two and three; it has a conserved H^1 norm and it possesses N -peakon solutions, when the polynomial has any degree; and it exhibits wave-breaking for certain solutions describing collisions between peakons and anti-peakons in the case $N = 2$.

1. INTRODUCTION

There is considerable interest in the study of equations of the form $u_t - u_{txx} = f(u, u_x, u_{xx}, u_{xxx})$ that describe breaking waves. In this paper we consider the equation

$$u_t - u_{txx} + au^p u_x - bu^{p-1} u_x u_{xx} - cu^p u_{xxx} = 0 \quad (1)$$

with parameters a, b, c (not all zero) and $p \neq 0$. This 4-parameter family contains several integrable equations. For $(p, a, b, c) = (1, 3, 2, 1)$ and $(p, a, b, c) = (1, 4, 3, 1)$, equation (1) reduces respectively to the Camassa-Holm equation [1]

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \quad (2)$$

and the Degasperis-Procesi equation [2]

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0 \quad (3)$$

while for $(p, a, b, c) = (2, 4, 3, 1)$, equation (1) becomes the Novikov equation [3]

$$u_t - u_{txx} + 4u^2 u_x - 3uu_x u_{xx} - u^2 u_{xxx} = 0. \quad (4)$$

The three equations (2), (3), (4) are integrable in the sense of having a Lax pair, a bi-Hamiltonian structure, as well as hierarchies of local symmetries and local conservation laws, and they also possess peaked travelling wave solutions.

In addition to these integrable equations, many other non-integrable equations that admit breaking waves are included in the 4-parameter family (1). For instance, there is the b -equation

$$u_t - u_{txx} + (b+1)uu_x - bu_x u_{xx} - uu_{xxx} = 0 \quad (5)$$

which unifies the Camassa-Holm and Degasperis-Procesi equations [4, 5]. There is also a modified version of the b -equation [6]

$$u_t - u_{txx} + (b+1)u^2u_x - buu_xu_{xx} - u^2u_{xxx} = 0 \quad (6)$$

which includes the Novikov equation. No other cases of the two equations (5) and (6) are known to be integrable [3, 4].

An equivalent form of the 4-parameter equation (1) is given by

$$m_t + \tilde{a}u^p u_x + bu^{p-1}u_x m + cu^p m_x = 0 \quad (7)$$

in terms of the momentum variable

$$m = u - u_{xx} \quad (8)$$

with parameters

$$\tilde{a} = a - b - c, \quad (\tilde{a}, b, c) \neq 0, \quad p \neq 0. \quad (9)$$

This parametric equation (7) is invariant under the group of scaling transformations $u \rightarrow \lambda u$, $t \rightarrow \lambda^s t$, $(\tilde{a}, b, c) \rightarrow \lambda^{s+p}(\tilde{a}, b, c)$ with $\lambda \neq 0$.

In section 2, we classify the low-order conservation laws of equation (1) and show that the Hamiltonians of the Camassa-Holm and Novikov equations are admitted as local conservation laws by equation (1) if and only if $\tilde{a} = 0$ and $b = p + 1$. We consider peaked travelling waves in section 3 and use a weak formulation of equation (1) to show that single peakon and multi-peakon solutions are admitted if and only if $\tilde{a} = 0$ and $c \neq 0$ when $p \geq 0$. We derive the explicit equations of motion for $N \geq 1$ peakon/anti-peakon solutions and also obtain the constants of motion inherited from the local conservation laws of equation (1).

In section 4, we combine the previous results to obtain a natural 1-parameter family of equations

$$m_t + (p+1)u^{p-1}u_x m + u^p m_x = 0, \quad p \geq 0 \quad (10)$$

given by $\tilde{a} = 0$, $b = p + 1$, $c \neq 0$, $p \geq 0$, where a scaling transformation $t \rightarrow t/c$ is used to put $c = 1$. Since this 1-parameter family (10) unifies the Camassa-Holm and Novikov equations, we will refer to it as the *gCHN equation*. (Similar unified equations have been considered previously from related perspectives [7, 8, 9, 10].) We then discuss some general features of the dynamics of its $N \geq 2$ peakon/anti-peakon solutions and we show that wave-breaking occurs for certain solutions describing collisions between peakons and anti-peakons in the case $N = 2$.

Finally, in section 5, we make some concluding remarks including a possible scenario for wave-breaking in the Cauchy problem for weak solutions.

2. CONSERVATION LAWS

For the 4-parameter equation (1), a *local conservation law* [11, 12] is a space-time divergence

$$D_t T + D_x X = 0 \quad (11)$$

holding for all solutions $u(t, x)$ of equation (1), where the *conserved density* T and the *spatial flux* X are functions of t , x , u and derivatives of u . The spatial integral of the conserved density T satisfies

$$\frac{d}{dt} \int_{-\infty}^{\infty} T dx = -X \Big|_{-\infty}^{\infty} \quad (12)$$

and so if the flux X vanishes at spatial infinity, then

$$\mathcal{C}[u] = \int_{-\infty}^{\infty} T dx = \text{const.} \quad (13)$$

formally yields a conserved quantity for equation (1). Conversely, any such conserved quantity arises from a local conservation law (11).

If the conserved quantity (13) is purely a boundary term, then the local conservation law is called *trivial*. This occurs when (and only when) the conserved density is a total x -derivative and the flux is a total t -derivative, related by

$$T = D_x \Theta, \quad X = -D_t \Theta \quad (14)$$

for all solutions $u(t, x)$ of equation (1), where Θ is some function of t, x, u and derivatives of u . Two local conservation laws are *equivalent* if they differ by a trivial conservation law, thereby giving the same conserved quantity up to boundary terms.

The set of all conservation laws (up to equivalence) admitted by equation (1) forms a vector space on which there is a natural action [11, 12, 13] by the group of all Lie symmetries of the equation.

For conserved densities and fluxes depending on at most $t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}$, a conservation law can be expressed in an equivalent form by a divergence identity

$$D_t T + D_x X = (u_t - u_{txx} + au^p u_x - bu^{p-1} u_x u_{xx} - cu^p u_{xxx}) Q \quad (15)$$

where

$$Q = -T_{u_{xx}} - X_{u_{tx}} \quad (16)$$

is called the *multiplier*. This identity (15)–(16) is called the *characteristic equation* [11, 12] for the conserved density and flux. By balancing the highest order t -derivative terms u_{ttt} on both sides of the equation, we directly find that $T_{u_{tt}} = 0$ and $X_{u_{tt}u_{tt}} = 0$. Then balancing the terms u_{tt} , we see that $X_{u_{tt}u_{tx}} = 0$. Hence the conserved density and the flux in the divergence identity must have the form

$$\begin{aligned} T &= T_0(t, x, u, u_t, u_x, u_{tx}, u_{xx}), \\ X &= X_0(t, x, u, u_t, u_x, u_{tx}, u_{xx}) + u_{tt} X_1(t, x, u, u_t, u_x, u_{xx}). \end{aligned} \quad (17)$$

Its multiplier (16) thus has the form

$$Q = Q_0(t, x, u, u_t, u_x, u_{tx}, u_{xx}). \quad (18)$$

In general, the differential order of a local conservation law is defined to be the smallest differential order among all equivalent conserved densities. A local conservation law is said to be of *low order* if the differential orders of T and X are both strictly less than the differential order of the equation.

Consequently, conserved densities and fluxes of the form (17) comprise all possible low-order conservation laws of equation (1). The problem of finding all low-order conservations then reduces to the simpler problem of finding all low-order multipliers (18). Since equation (1) is an evolution equation, it has no Lagrangian formulation in terms of the variable u . In this situation, the problem of finding multipliers can be understood as a kind of adjoint [14] of the problem of finding symmetries.

An *infinitesimal symmetry* [11, 15, 12] of equation (1) is a generator

$$\hat{X} = P \partial_u \quad (19)$$

whose coefficient P is given by a function of t, x, u and derivatives of u , such that the prolonged generator satisfies the invariance condition

$$\begin{aligned} 0 &= \text{pr}\hat{X}(u_t - u_{txx} + au^p u_x - bu^{p-1} u_x u_{xx} - cu^p u_{xxx}) \\ &= D_t P - D_t D_x^2 P + a D_x(u^p P) - bu^{p-1} D_x(u_x D_x P) - cu^p D_x^3 P \\ &\quad - (b(p-1)u^{p-2} u_x u_{xx} + cpu^{p-1} u_{xxx})P \end{aligned} \quad (20)$$

holding for all solutions $u(t, x)$ of equation (1). The Lie symmetry group of equation (1) is generated by infinitesimal symmetries (19) with coefficients of the form

$$P(t, x, u, u_t, u_x). \quad (21)$$

If P is at most linear in u_t and u_x , then the resulting generator (19) will yield a group of point transformations [11, 15], whereas if P is nonlinear in u_t or u_x , then a group of contact transformations [11, 15] will be generated. Hence, all generators of Lie symmetries admitted by equation (1) are determined by the solutions of condition (20) for $P(t, x, u, u_t, u_x)$. (It is straightforward to solve this determining equation by Maple to classify the Lie symmetry group of equation (1), as shown in the Appendix.)

The condition for determining all multipliers $Q(t, x, u, u_t, u_x, u_{tx}, u_{xx})$ of low-order conservation laws (18) admitted by equation (1) consists of

$$E_u((u_t - u_{txx} + au^p u_x - bu^{p-1} u_x u_{xx} - cu^p u_{xxx})Q) = 0 \quad (22)$$

which arises from the property that the variational derivative (Euler operator)

$$E_u = \partial_u - D_x \partial_{u_x} - D_t \partial_{u_t} + D_x^2 \partial_{u_{xx}} + D_t^2 \partial_{u_{tt}} + D_x D_t \partial_{u_{tx}} - \dots \quad (23)$$

annihilates an expression identically iff it is a space-time divergence [11, 12]. This condition (22) can be split with respect to u_{txx} and t, x -derivatives of u_{txx} , yielding an equivalent overdetermined system of equations on Q . One equation in this system is given by the adjoint of the symmetry determining equation (20),

$$\begin{aligned} 0 &= -D_t Q + D_t D_x^2 Q - au^p D_x Q - bD_x(u_x D_x(u^{p-1} Q)) + cD_x^3(u^p Q) \\ &\quad - (b(p-1)u^{p-2} u_x u_{xx} + cpu^{p-1} u_{xxx})Q \end{aligned} \quad (24)$$

holding for all solutions $u(t, x)$ of equation (1). Solutions Q of this equation (24) are called *adjoint-symmetries* (or *cosymmetries*) [14, 15, 16, 17]. The remaining equations in the system comprise Helmholtz conditions which are necessary and sufficient for Q to have the form (16). As a consequence, multipliers (18) are simply adjoint-symmetries that have a certain variational form.

For any solution (18) of the multiplier determining equation (22), a corresponding conserved density and flux of the form (17) can be recovered either through integration [12] of the characteristic equation (15), which splits with respect to $u_{ttx}, u_{txx}, u_{xxx}, u_{tt}$ into a system of equations for T and X , or through a homotopy integral formula [12, 11, 18, 19], which expresses T and X directly in terms of $(u_t - u_{txx} + au^p u_x - bu^{p-1} u_x u_{xx} - cu^p u_{xxx})Q$. It is straightforward to show that T and X have the form (14) of a trivial conservation law iff $Q = 0$. Thus there is a one-to-one correspondence between equivalence classes of non-trivial low-order conservation laws (17) and non-zero low-order multipliers (18).

2.1. **Classification results.** Both the Camassa-Holm equation (2) and Novikov equation (4) possess low-order local conservation law given by the conserved densities [1, 20, 21]

$$T = mu = u^2 + u_x^2 + D_x(-uu_x), \quad (25)$$

$$T = m^q, \quad (26)$$

where $q = 1/2$ and $q = 2/3$, respectively, for the two equations. In addition, the Camassa-Holm equation (2) itself is a low-order local conservation law having the conserved density

$$T = m = u + D_x(-u_x). \quad (27)$$

All of these conserved densities are related to Hamiltonian structures for the two equations [1, 20, 21]. The corresponding multipliers are respectively given by

$$Q = u, \quad (28)$$

$$Q = qm^{q-1}, \quad (29)$$

$$Q = 1. \quad (30)$$

To look for conserved densities of the same form for equation (1), we now classify all multipliers up to 1st-order

$$Q = Q(t, x, u, u_x, u_t) \quad (31)$$

as well as all 2nd-order multipliers with the specific form

$$Q = Q(u, u_x, u_{xx}). \quad (32)$$

In each case it is straightforward to solve the determining equation (22) by use of Maple (as shown in the Appendix), which leads to the following classification result.

Proposition 2.1. (i) Equation (1) admits 0th-order multipliers only in the following cases:

$$(a) \quad Q = 1 \quad \text{iff} \quad p = 1 \quad \text{or} \quad b = pc \quad (33)$$

$$(b) \quad Q = u \quad \text{iff} \quad b = (p + 1)c \quad (34)$$

$$(c) \quad Q = \exp(\pm \sqrt{a/c} x) \quad \text{iff} \quad p = 1, \quad b = 3c \quad (35)$$

$$(d) \quad Q = f(t) \exp(\pm x) \quad \text{iff} \quad p = 1, \quad a = c, \quad b = 3c \quad (36)$$

$$(e) \quad Q = x - ctu \quad \text{iff} \quad p = 1, \quad a = c, \quad b = 2c \quad (37)$$

(ii) For any $p \neq 0$ and any $(a, b, c) \neq 0$, equation (1) admits no 1st-order multipliers.

(iii) Equation (1) admits 2nd-order multipliers of the form (32) only in the following cases:

$$(a) \quad Q = (u - u_{xx})^{q-1} \quad \text{iff} \quad q = pc/b \neq 1, \quad a = b + c \quad (38)$$

$$(b) \quad Q = 2au - (p + 2)cu_{xx} \quad \text{iff} \quad b = \frac{1}{2}pc, \quad c \neq 0, \quad p \neq -2 \quad (39)$$

In light of the adjoint relationship between multipliers and symmetries, the classification of 0th- and 1st- order multipliers in Proposition 2.1 is a counterpart of the classification of Lie symmetries (cf. Proposition A.1).

Next we obtain the corresponding conserved densities and fluxes for each multiplier (33)–(38) by first splitting the characteristic equation (15) with respect to u_{ttx} , u_{txx} , u_{xxx} , u_{tt} where T and X have the form (17), and then integrating the resulting system of equations. This yields the following low-order local conservation laws for equation (1).

Theorem 2.1. (i) *The local conservation laws admitted by the wave-breaking equation (1) with multipliers of at most 1st-order consist of three 0th-order conservation laws*

$$T_1 = u, \quad X_1 = \frac{a}{p+1}u^{p+1} + \frac{1}{2}(pc - b)u_x^2 - cu^p u_{xx} + u_{tx} \quad (40)$$

iff $p = 1$ or $b = pc$;

$$T_2 = (c - a)e^{\pm\sqrt{a/c}x}u, \quad X_2 = e^{\pm\sqrt{a/c}x}(\pm\sqrt{ac}(u_t + cuu_x) - cu_{tx} - c^2(u_x^2 + uu_{xx})) \quad (41)$$

iff $p = 1$, $b = 3c$ ($c \neq 0$);

$$T_3 = 0, \quad X_3 = f(t)e^{\pm x}(\pm(u_t + cuu_x) - u_{tx} - c(u_x^2 + uu_{xx})) \quad (42)$$

iff $p = 1$, $a = c$, $b = 3c$;

and two 1st-order conservation laws

$$T_4 = \frac{1}{2}(u^2 + u_x^2), \quad X_4 = \left(\frac{a}{p+2}u - cu_{xx}\right)u^{p+1} - uu_{tx} \quad (43)$$

iff $b = (p+1)c$;

$$T_5 = -\frac{1}{2}ct(u^2 + u_x^2) + xu, \quad X_5 = (ctu - x)(u_{tx} + cuu_{xx}) + u_t - \frac{1}{3}c^2tu^3 + \frac{1}{2}cx(u^2 - u_x^2 + 2uu_x) \quad (44)$$

iff $p = 1$, $a = c$, $b = 2c$.

(ii) *The local conservation laws admitted by the wave-breaking equation (1) with 2nd-order multipliers of the form (32) consist of two 2nd-order conservation laws*

$$T_6 = (u - u_{xx})^{pc/b}, \quad X_6 = cu^p(u - u_{xx})^{pc/b}, \quad (45)$$

iff $a = b + c$ ($b \neq pc$, $c \neq 0$);

$$T_7 = au^2 + (a + b + c)u_x^2 + (b + c)u_{xx}^2, \quad X_7 = \frac{2}{p+2}(au - (b + c)u_{xx})^2u^p - 2a uu_{tx} - 2(b + c)u_t u_x \quad (46)$$

iff $b = \frac{1}{2}pc$ ($c \neq 0$).

In these conservation laws (40)–(46), any terms of the form $q^{-1}u^q$ in the case $q = 0$ should be replaced by $\ln|u|$.

These conservation laws yield the following conserved integrals. We start with the conservation laws at 0th order. From T_1 , we have

$$\mathcal{C}_1 = \int_{-\infty}^{\infty} u \, dx, \quad p = 1 \quad \text{or} \quad b = pc \quad (47)$$

which is the conserved mass for equation (1). The conserved integral arising from T_2 is a weighted mass,

$$\mathcal{C}_2 = \int_{-\infty}^{\infty} e^{\pm\sqrt{a/c}x} u \, dx, \quad p = 1, \quad b = 3c \neq 0 \quad (c \neq a). \quad (48)$$

Interestingly, from T_3 we get a conserved integral which vanishes, but has a non-zero spatial flux. This type of conservation law arises because the multiplier (36) converts equation (1) into the form of a total x -derivative.

Next we look at the conservation laws at 1st order. From T_4 , the H^1 norm of $u(t, x)$ is conserved,

$$\mathcal{C}_4 = \int_{-\infty}^{\infty} u^2 + u_x^2 \, dx = \|u\|_{H^1}, \quad b = (p+1)c. \quad (49)$$

From T_5 , we have

$$\begin{aligned} \mathcal{C}_5 &= t \int_{-\infty}^{\infty} u^2 + u_x^2 \, dx - \frac{2}{c} \int_{-\infty}^{\infty} xu \, dx \\ &= t\|u\|_{H^1} - \frac{2}{c}\mathcal{P}(t), \quad p = 1, \quad a = c, \quad b = 2c \neq 0 \end{aligned} \quad (50)$$

where $\mathcal{P}(t) = \int_{-\infty}^{\infty} xu \, dx$ is the center of mass of $u(t, x)$. Since \mathcal{C}_5 is conserved, it can be evaluated at $t = 0$, which yields the relation $\mathcal{P}(t) = \mathcal{P}(0) + (c/2)t\|u\|_{H^1}$. This shows that the center of mass moves at a constant speed controlled by the H^1 norm of $u(t, x)$.

Finally, we consider the conservation laws at 2nd order. From T_6 , we get

$$\mathcal{C}_6 = \int_{-\infty}^{\infty} (u - u_{xx})^q \, dx, \quad a = b + c, \quad q = pc/b \neq 1. \quad (51)$$

This shows that the L^q norm of $m = u - u_{xx}$ is conserved if m does not change sign or if q is an even integer. The conserved integral arising from T_7 is a linear combination of the L^2 norms of u, u_x, u_{xx} as given by

$$\begin{aligned} \mathcal{C}_7 &= \int_{-\infty}^{\infty} au^2 + (a+b+c)u_x^2 + (b+c)u_{xx}^2 \, dx \\ &= a\|u\|_{L^2} + (a+b+c)\|u_x\|_{L^2} + (b+c)\|u_{xx}\|_{L^2}, \quad b = \frac{1}{2}pc \neq 0. \end{aligned} \quad (52)$$

This can be written alternatively as a weighted H^2 norm when $b+c \neq 0$. It is interesting to note that simultaneous conservation of both the H^1 and the weighted H^2 norms requires the condition $b = (p+1)c = \frac{1}{2}pc$ which holds iff $p = -2$ and $b+c = 0$, but in this case $\mathcal{C}_7 = a\mathcal{C}_4 = a\|u\|_{H^1}$.

3. PEAKON SOLUTIONS

Both the Camassa-Holm and Novikov equations possess peaked travelling wave solutions [1, 20], called peakons,

$$u(t, x) = v^q \exp(-|x - vt|), \quad v = \text{const}. \quad (53)$$

where $q = 1$ and $q = 1/2$, respectively, for the two equations. Peakons have attracted much attention in the study of breaking wave equations.

In general, on $-\infty < x < \infty$, a peakon is a weak travelling wave solution satisfying an integral (i.e. weak) formulation of a breaking wave equation. Such a formulation is essential

for deriving multi-peakon solutions. However, single peakons can be derived directly from the travelling wave reduction of a breaking wave equation, which will be the approach we use here.

3.1. Single peakon solution. The manifest invariance of the 4-parameter equation (1) under time-translation and space-translation symmetries implies the existence of travelling wave solutions

$$u = \phi(z), \quad z = x - vt, \quad v = \text{const.} \neq 0 \quad (54)$$

where $\phi(z)$ satisfies the ODE

$$-v(\phi - \phi'')' + a\phi^p\phi' - b\phi^{p-1}\phi'\phi'' - c\phi^p\phi''' = 0. \quad (55)$$

For the travelling wave ODE (55), an integral formulation is obtained through multiplying this ODE by a test function ψ (which is smooth and has compact support) and integrating over $-\infty < z < \infty$, leaving at most first derivatives of ϕ in the integral, which yields

$$0 = \int_{-\infty}^{+\infty} \left(v(\psi'' - \psi)\phi' + (a\psi - c\psi'')\phi^p\phi' + \frac{1}{2}(b - 3pc)\psi'\phi^{p-1}\phi'^2 + \frac{1}{2}(p-1)(b - pc)\psi\phi^{p-2}\phi'^3 \right) dz. \quad (56)$$

A weak solution of ODE (55) is a function $\phi(z)$ that belongs to the Sobolev space $W_{\text{loc}}^{1,3}(\mathbb{R})$ and that satisfies the integral equation (56) for all smooth test functions $\psi(z)$ with compact support on \mathbb{R} .

To proceed we substitute a peaked travelling wave expression

$$\phi = \alpha e^{-|z|}, \quad \alpha = \text{const.} \quad (57)$$

into equation (56) and split up the integral into the intervals $(-\infty, 0)$ and $(0, +\infty)$. The first term in equation (56) yields, after integration by parts,

$$\int_{-\infty}^0 v(\psi'' - \psi)\phi' dz + \int_0^{+\infty} v(\psi'' - \psi)\phi' dz = 2\alpha v\psi'(0). \quad (58)$$

Similarly, the second term in equation (56) gives

$$\begin{aligned} & \int_{-\infty}^0 (a\psi - c\psi'')\phi^p\phi' dz + \int_0^{+\infty} (a\psi - c\psi'')\phi^p\phi' dz \\ &= -2\alpha^{p+1}c\psi'(0) + \alpha^{p+1}(-a + (p+1)^2c) \int_{-\infty}^{+\infty} \text{sgn}(z)\psi e^{-(p+1)|z|} dz \end{aligned} \quad (59)$$

provided $p+1 > 0$ so that the boundary terms at $z = \pm\infty$ vanish. The third and fourth terms in equation (56) together yield

$$\begin{aligned} & \int_{-\infty}^0 \left(\frac{1}{2}(p-1)(b - pc)\psi\phi^{p-2}\phi'^3 + \frac{1}{2}(b - 3pc)\psi'\phi^{p-1}\phi'^2 \right) dz \\ &+ \int_0^{+\infty} \left(\frac{1}{2}(p-1)(b - pc)\psi\phi^{p-2}\phi'^3 + \frac{1}{2}(b - 3pc)\psi'\phi^{p-1}\phi'^2 \right) dz \\ &= \alpha^{p+1}(b - p(p+2)c) \int_{-\infty}^{+\infty} \text{sgn}(z)\psi e^{-(p+1)|z|} dz. \end{aligned} \quad (60)$$

When the terms (58)–(60) are combined, we find that equation (56) reduces to

$$0 = 2\alpha(v - c\alpha^p)\psi'(0) + \alpha^{p+1}(b + c - a) \int_{-\infty}^{+\infty} \operatorname{sgn}(z)\psi e^{-(p+1)|z|} dz. \quad (61)$$

This equation is satisfied for all test functions ψ iff

$$a = b + c, \quad c\alpha^p = v, \quad (62)$$

which determines the amplitude α in the peakon expression (57). Thus we obtain the following result.

Proposition 3.1. *The travelling wave equation (56) admits a peakon solution only in the case*

$$\phi(z) = (v/c)^{1/p} e^{-|z|}, \quad a = b + c, \quad c \neq 0, \quad p + 1 > 0 \quad (63)$$

where $v = \text{const.}$ is the wave speed.

The resulting peakon solution of equation (1) is given by

$$u(t, x) = c^{-1/p} v^{1/p} \exp(-|x - vt|), \quad a = b + c. \quad (64)$$

When the nonlinearity power p is a positive integer, then the wave speed is necessarily positive, $v > 0$, if p is even, as in the case ($p = 2$) of the Novikov equation (4), while if p is odd, the wave speed can be either positive or negative, $v \gtrless 0$, as in the case ($p = 1$) of the Camassa-Holm equation (2).

The peakon solution (64) satisfies equation (1) only in the sense of a weak solution. This means $u(t, x)$ is a distribution in $L_{\text{loc}}^{\infty}(-T, T)$ with respect to $t \in (-T, T)$ for some $T > 0$ and in $W_{\text{loc}}^{1,3}(\mathbb{R})$ with respect to $x \in \mathbb{R}$ such that it satisfies the integral equation

$$0 = \iint_{-\infty}^{+\infty} \left((\psi - \psi_{xx})u_t + (a\psi - c\psi_{xx})u^p u_x + \frac{1}{2}(b - 3pc)\psi_x u^{p-1} u_x^2 + \frac{1}{2}(p-1)(b - pc)\psi u^{p-2} u_x^3 \right) dx dt \quad (65)$$

for all test functions $\psi(t, x)$ in $C_0^{\infty}((-T, T) \times \mathbb{R})$.

3.2. Multi-peakon solution. Both the Camassa-Holm and Novikov equations possess multi-peakon solutions [1, 20, 22] which are a linear superposition of peaked travelling waves with time-dependent amplitudes and positions. The form of these solutions is given by

$$u(t, x) = \sum_{i=1}^N \alpha_i(t) \exp(-|x - \beta_i(t)|), \quad N = 1, 2, \dots \quad (66)$$

where the amplitudes $\alpha_i(t)$ and positions $\beta_i(t)$ satisfy a Hamiltonian system of ODEs

$$\alpha_i' = \{\alpha_i, H\}, \quad \beta_i' = \{\beta_i, H\}, \quad i = 1, \dots, N \quad (67)$$

given in terms of the Hamiltonian function

$$H = \frac{1}{2} \sum_{j,k=1}^N \alpha_j \alpha_k \exp(-|\beta_j - \beta_k|). \quad (68)$$

The Poisson bracket $\{f, g\}$ in this system (67) arises from the respective Hamiltonian operator formulations [1, 20] of these two equations and has the standard canonical form in the

case of Camassa-Holm equation and a certain non-canonical form in the case of the Novikov equation.

We now investigate whether equation (1) also admits multi-peakon solutions. It will be convenient to use the notation

$$u = \sum_i \alpha_i e^{-|z_i|}, \quad z_i = x - \beta_i \quad (69)$$

where the summation is understood to go from 1 to N . Note that the x -derivatives of u are given by

$$u_x = - \sum_i \operatorname{sgn}(z_i) \alpha_i e^{-|z_i|} \quad (70)$$

and

$$u_{xx} = \sum_i (-2\delta(z_i) + \operatorname{sgn}(z_i)^2) \alpha_i e^{-|z_i|} \quad (71)$$

in terms of the sign function

$$\operatorname{sgn}(z) = \begin{cases} 1 & z > 0 \\ -1 & z < 0 \\ 0 & z = 0 \end{cases} \quad (72)$$

and the Dirac delta distribution

$$\delta(z) = \frac{d(\frac{1}{2}\operatorname{sgn}(z))}{dz} \quad (73)$$

which has the properties $\delta(z) = 0$ for $z \neq 0$, and $\int_{-\epsilon}^{\epsilon} \delta(z) dz = 1$ for all $\epsilon > 0$.

To begin, we substitute the general multi-peakon expression (69) into the integral equation (65). There are two ways we can then proceed. One way is to assume $\beta_1 < \beta_2 < \dots < \beta_N$ at a fixed $t > 0$, split up the integral over x into corresponding intervals, and integrate by parts, similarly to the derivation of the single peakon solution. Another way, which is simpler, is to employ the following result from distribution theory [23].

Let $f(x)$ be a piecewise C^1 function having at most jump discontinuities at a finite number of points $x = x_i$ in \mathbb{R} . Then, for any test function $\psi(x)$,

$$\int_{-\infty}^{\infty} \psi' f dx = - \sum_i \psi(x_i) [f]_{x_i} - \int_{-\infty}^{\infty} \psi \langle f' \rangle dx \quad (74)$$

where

$$[f]_{x_i} = f(x_i^+) - f(x_i^-) \quad (75)$$

is the jump in $f(x)$ at the point $x = x_i$, and

$$\langle f' \rangle = \begin{cases} f' & x \neq x_i \\ 0 & x = x_i \end{cases} \quad (76)$$

is the non-singular part of the distributional derivative of $f(x)$. We will now use this integration by parts relation (74) to evaluate each term in the integral equation (65).

The first term in equation (65) yields

$$\begin{aligned} \iint_{-\infty}^{+\infty} (\psi u_t - \psi_{xx} u_t) dx dt &= \iint_{-\infty}^{+\infty} \psi(u_t - \langle u_{txx} \rangle) dx dt \\ &+ \sum_i \int_{-\infty}^{+\infty} (\psi_x(t, \beta_i)[u_t]_{\beta_i} - \psi(t, \beta_i)[u_{tx}]_{\beta_i}) dt. \end{aligned} \quad (77)$$

From expression (71), we see

$$\langle u_{xx} \rangle = \sum_i \operatorname{sgn}(z_i)^2 \alpha_i e^{-|z_i|} = u \quad \text{for } x \neq \beta_i, \quad (78)$$

so thus $u_t - \langle u_{txx} \rangle = 0$ holds a. e. in $(-\infty, \infty)$. Hence

$$\iint_{-\infty}^{+\infty} \psi(u_t - \langle u_{txx} \rangle) dx dt = 0 \quad (79)$$

and thus we get

$$\iint_{-\infty}^{+\infty} (\psi u_t - \psi_{xx} u_t) dx dt = \sum_i \int_{-\infty}^{+\infty} (\psi_x(t, \beta_i)[u_t]_{\beta_i} - \psi(t, \beta_i)[u_{tx}]_{\beta_i}) dt. \quad (80)$$

Next, the second term in equation (65) gives

$$\begin{aligned} \iint_{-\infty}^{+\infty} (a\psi u^p u_x - c\psi_{xx} u^p u_x) dx dt &= \iint_{-\infty}^{+\infty} (a\psi u^p u_x + c\psi_x(u^p \langle u_{xx} \rangle + pu^{p-1} \langle u_x^2 \rangle)) dx dt \\ &+ \sum_i \int_{-\infty}^{+\infty} \psi_x(t, \beta_i)[u^p u_x]_{\beta_i} dt. \end{aligned} \quad (81)$$

We now simplify the two parts of the integral involving ψ_x . For the first part, we have

$$\iint_{-\infty}^{+\infty} \psi_x u^p \langle u_{xx} \rangle dx dt = \iint_{-\infty}^{+\infty} \psi_x u^{p+1} dx dt = - \iint_{-\infty}^{+\infty} \psi(p+1)u^p u_x dx dt \quad (82)$$

after using relation (78) and then integrating by parts. For the second part, since $\langle u_x^2 \rangle = u_x^2$ holds a. e. in $(-\infty, \infty)$, we have

$$\begin{aligned} \iint_{-\infty}^{+\infty} \psi_x u^{p-1} \langle u_x^2 \rangle dx dt &= \iint_{-\infty}^{+\infty} \psi_x u^{p-1} u_x^2 dx dt \\ &= \sum_i \int_{-\infty}^{+\infty} \psi(t, \beta_i)[u^{p-1} u_x^2]_{\beta_i} dt - \iint_{-\infty}^{+\infty} \psi(u^{p-1} \langle u_x^2 \rangle)_x dx dt \end{aligned} \quad (83)$$

from applying the integration by parts relation (74). By simplifying $(u^{p-1} \langle u_x^2 \rangle)_x = (p-1)u^{p-2} \langle u_x^3 \rangle + 2u^{p-1} \langle u_x u_{xx} \rangle = (p-1)u^{p-2} u_x^3 + 2u^p u_x$ a. e. with the use of relation (78), we see

$$\iint_{-\infty}^{+\infty} \psi(u^{p-1} \langle u_x^2 \rangle)_x dx dt = \iint_{-\infty}^{+\infty} \psi((p-1)u^{p-2} u_x^3 + 2u^p u_x) dx dt. \quad (84)$$

Hence the integral (83) becomes

$$\begin{aligned} \iint_{-\infty}^{+\infty} \psi_x u^{p-1} \langle u_x^2 \rangle dx dt &= - \iint_{-\infty}^{+\infty} \psi ((p-1)u^{p-2}u_x^3 + 2u^p u_x) dx dt \\ &\quad - \sum_i \int_{-\infty}^{+\infty} \psi(t, \beta_i) u(\beta_i)^{p-1} [u_x^2]_{\beta_i} dt. \end{aligned} \quad (85)$$

Then we have

$$\begin{aligned} &\iint_{-\infty}^{+\infty} (a\psi u^p u_x - c\psi_{xx} u^p u_x) dx dt \\ &= \iint_{-\infty}^{+\infty} \psi ((a - (3p+1)c)u^p u_x - cp(p-1)u^{p-2}u_x^3) dx dt \\ &\quad - \sum_i \int_{-\infty}^{+\infty} cp\psi(t, \beta_i) u(\beta_i)^{p-1} [u_x^2]_{\beta_i} dt + \sum_i \int_{-\infty}^{+\infty} c\psi_x(t, \beta_i) u(\beta_i)^p [u_x]_{\beta_i} dt. \end{aligned} \quad (86)$$

Similarly, the third term in equation (65) gives

$$\begin{aligned} \iint_{-\infty}^{+\infty} \frac{1}{2}(b - 3pc)\psi_x u^{p-1} u_x^2 dx dt &= \iint_{-\infty}^{+\infty} (3pc - b)\psi (u^{p-1} \langle u_x u_{xx} \rangle + \frac{1}{2}(p-1)u^{p-2} \langle u_x^3 \rangle) dx dt \\ &\quad + \sum_i \int_{-\infty}^{+\infty} \frac{1}{2}(3pc - b)\psi(t, \beta_i) [u^{p-1} u_x^2]_{\beta_i} dt. \end{aligned} \quad (87)$$

We can simplify the integral involving ψ by the same steps used for the previous integral. This yields

$$\iint_{-\infty}^{+\infty} \psi (u^{p-1} \langle u_x u_{xx} \rangle + \frac{1}{2}(p-1)u^{p-2} \langle u_x^3 \rangle) dx dt = \iint_{-\infty}^{+\infty} \psi (u^p u_x + \frac{1}{2}(p-1)u^{p-2} u_x^3) dx dt. \quad (88)$$

Hence we then have

$$\begin{aligned} \iint_{-\infty}^{+\infty} \frac{1}{2}(b - 3pc)\psi_x u^{p-1} u_x^2 dx dt &= \iint_{-\infty}^{+\infty} (3pc - b)\psi (u^p u_x + \frac{1}{2}(p-1)u^{p-2} u_x^3) dx dt \\ &\quad + \sum_i \int_{-\infty}^{+\infty} \frac{1}{2}(3pc - b)\psi(t, \beta_i) u(\beta_i)^{p-1} [u_x^2]_{\beta_i} dt. \end{aligned} \quad (89)$$

Finally, by combining the three terms (80), (86), (89) with the fourth term in equation (65), we obtain

$$\begin{aligned} 0 &= (a - b - c) \iint_{-\infty}^{+\infty} u^p u_x \psi dx dt + \sum_i \int_{-\infty}^{+\infty} \psi_x(t, \beta_i) ([u_t]_{\beta_i} + cu(t, \beta_i)^p [u_x]_{\beta_i}) dt \\ &\quad + \sum_i \int_{-\infty}^{+\infty} \psi(t, \beta_i) (-[u_{tx}]_{\beta_i} + \frac{1}{2}(pc - b)u(t, \beta_i)^{p-1} [u_x^2]_{\beta_i}) dt. \end{aligned} \quad (90)$$

The jump terms are evaluated by

$$[u_t]_{\beta_i} = 2\alpha_i\beta_i', \quad [u_x]_{\beta_i} = -2\alpha_i \quad (91)$$

$$[u_{tx}]_{\beta_i} = \frac{d[u_x]_{\beta_i}}{dt} = -2\alpha_i' \quad (92)$$

$$[u_x^2]_{\beta_i} = 2u_x(\beta_i)[u_x]_{\beta_i} = -4\alpha_i u_x(\beta_i) \quad (93)$$

which all follow directly from the expressions (69) and (70). Thus we get

$$\begin{aligned} 0 = & (a - b - c) \iint_{-\infty}^{+\infty} u^p u_x \psi dx dt + 2 \sum_i \int_{-\infty}^{+\infty} \psi_x(t, \beta_i) (\alpha_i \beta_i' - c \alpha_i u(t, \beta_i)^p) dt \\ & + 2 \sum_i \int_{-\infty}^{+\infty} \psi(t, \beta_i) (\alpha_i' + (b - pc) \alpha_i u(t, \beta_i)^{p-1} u_x(t, \beta_i)) dt. \end{aligned} \quad (94)$$

This equation is satisfied for all test functions ψ iff

$$a = b + c, \quad p \geq 0, \quad \beta_i' = c u(t, \beta_i)^p, \quad \alpha_i' = (pc - b) \alpha_i u(t, \beta_i)^{p-1} u_x(t, \beta_i) \quad (95)$$

which determines the amplitudes α_i and the positions β_i in the multi-peakon expression (69). Thus we have established the following result.

Proposition 3.2. *The integral equation (65) admits an N -peakon solution (69) for all $N \geq 1$ only in the case*

$$a = b + c, \quad p \geq 0. \quad (96)$$

From Propositions 3.2 and 3.1, we have a classification of all cases for which the 4-parameter equation (1) possesses both single peakon and multi-peakon solutions.

Theorem 3.1. *The 4-parameter equation (1) admits single peakon and multi-peakon solutions iff*

$$a = b + c, \quad c \neq 0, \quad p \geq 0. \quad (97)$$

In this case, a general N -peakon solution has the form (66), where the amplitudes $\alpha_i(t)$ and positions $\beta_i(t)$ satisfy the system of ODEs

$$\beta_i' = c \left(\alpha_j + \sum_{\substack{j \neq i \\ j=1}}^N \alpha_j \exp(-|\beta_{i,j}|) \right)^p, \quad (98)$$

$$\alpha_i' = (b - pc) \alpha_i \left(\alpha_j + \sum_{\substack{j \neq i \\ j=1}}^N \alpha_j \exp(-|\beta_{i,j}|) \right)^{p-1} \sum_{\substack{k \neq i \\ k=1}}^N \operatorname{sgn}(\beta_{i,k}) \alpha_k \exp(-|\beta_{i,k}|), \quad (99)$$

in terms of the separations

$$\beta_{i,j} = \beta_i - \beta_j. \quad (100)$$

This result generalizes related work in Ref.[8] which established the existence of single and multi-peakon solutions for a 2-parameter equation defined by the case $a = b + 1$, $c = 1$ of equation (7). (In particular, the derivation in Ref.[8] was completely formal, whereas the steps here provide a rigorous proof applied to the more general 4-parameter equation (7).)

It is easy to check that the general N -peakon ODE system (98)–(99) reduces to the well-known multi-peakon systems for the b -equation (5) when $(p, a, b, c) = (1, b + 1, b, 1)$, which

includes the the Camassa-Holm equation and the Degasperis-Procesi equation when $b = 2$ and $b = 3$, respectively, as well as for the Novikov equation when $(p, a, b, c) = (2, 4, 3, 1)$.

3.3. Constants of motion. The ODE system (98)–(99) for the amplitudes and positions of the N peakons in the expression (66) inherits constants of motion (i.e. time-independent quantities) given by the conserved integrals that are admitted by equation (7) in the case (97). From Theorem 2.1, there are six conserved integrals (47)–(52) which we can consider.

The first conserved integral (47) yields

$$\begin{aligned} \mathcal{C}_1 &= \int_{-\infty}^{\infty} u \, dx = 2 \sum_{i=1}^N \alpha_i = \text{const.} \\ \text{iff } &p = 1 \quad \text{or} \quad b = pc. \end{aligned} \tag{101}$$

This quantity \mathcal{C}_1 is the total mass for the N -peakon solution. A weighted mass arises from the second conserved integral (48),

$$\begin{aligned} \mathcal{C}_2 &= \int_{-\infty}^{\infty} e^{-\sqrt{a/c}|x|} u \, dx = \frac{2c}{c-a} \sum_{i=1}^N (e^{-\sqrt{a/c}|\beta_i|} - e^{-|\beta_i|}) \alpha_i = \text{const.} \\ \text{iff } &p = 1, \quad a = 4c, \quad b = 3c. \end{aligned} \tag{102}$$

The next conserved integral (49) gives

$$\begin{aligned} \mathcal{C}_4 &= \int_{-\infty}^{\infty} u^2 + u_x^2 \, dx = \int_{-\infty}^{\infty} u(u - u_{xx}) \, dx = 2 \sum_{i,j=1}^N \alpha_i \alpha_j e^{-|\beta_i - \beta_j|} = \text{const.} \\ \text{iff } &a = (p+2)c, \quad b = (p+1)c \end{aligned} \tag{103}$$

which is the H^1 norm of the N -peakon solution.

The fourth conserved integral (50) does not exist in the case (97), since $a = b + c = c$ and $b = 2c$ together imply that $a = b = c = 0$. Last, the two conserved integrals (51) and (52) are nonlinear in u_{xx} which is a distribution. As a consequence, both these integrals are ill-defined for the the N -peakon solution.

4. UNIFIED FAMILY OF CAMASSA-HOLM-NOVIKOV EQUATIONS

From Theorems 2.1 and 3.1, the low-order conservation laws (25)–(26) as well as the N -peakon solution expression (66) of the Camassa-Holm and Novikov equations are admitted simultaneously by the 4-parameter equation (7) iff its parameters (\tilde{a}, b, c, p) satisfy

$$\tilde{a} = 0, \quad b = (p+1)c, \quad c \neq 0, \quad p \geq 0. \tag{104}$$

After a scaling transformation $t \rightarrow t/c$ is used to put $c = 1$, equation (7) reduces to the 1-parameter gCHN equation (10) presented in section 1.

4.1. Dynamics of multi-peakon solutions. The explicit system describing N -peakon solutions of the gCHN equation (10) for all $p \geq 0$ is given by

$$\beta_i' = \left(\alpha_j + \sum_{\substack{j \neq i \\ j=1}}^N \alpha_j \exp(-|\beta_{i,j}|) \right)^p, \quad \beta_{i,j} = \beta_i - \beta_j, \quad (105)$$

$$\alpha_i' = \alpha_i \left(\alpha_j + \sum_{\substack{j \neq i \\ j=1}}^N \alpha_j \exp(-|\beta_{i,j}|) \right)^{p-1} \sum_{\substack{k \neq i \\ k=1}}^N \operatorname{sgn}(\beta_{i,k}) \alpha_k \exp(-|\beta_{i,k}|), \quad (106)$$

where $\alpha_i(t)$ and $\beta_i(t)$ are, respectively, the amplitudes and positions appearing in the general N -peakon expression (66). The H^1 norm (103) of the N -peakon solution provides a constant of motion

$$H = \sum_{i,j=1}^N \alpha_i \alpha_j e^{-|\beta_{i,j}|} = \operatorname{const.} \geq 0 \quad (107)$$

which is determined by the initial amplitudes and initial separations.

When all of the amplitudes are positive, $\alpha_i > 0$, for all $t \geq 0$, the solution expression (66) is a superposition of $N \geq 1$ peakons, each of which is right moving. In this case, the constant of motion (107) directly gives the inequality $H > \alpha_i > 0$, $i = 1, 2, \dots, N$, which implies that any collisions among the N peakons are elastic.

When all of the amplitudes are negative, $\alpha_i < 0$, for all $t \geq 0$, the solution expression (66) is instead a superposition of $N \geq 1$ anti-peakons, each of which is either right moving if p is even or left moving if p is odd. Similarly to the previous case, the constant of motion (107) yields $H > |\alpha_i| > 0$, $i = 1, 2, \dots, N$, implying that any collisions among the N anti-peakons are elastic.

In the case when some amplitudes have opposite signs, or an amplitude changes its sign at some $t > 0$, the solution expression (66) then describes a superposition of both peakons and anti-peakons. Although the constant of motion is still non-negative, the amplitudes are no longer bounded by $H \geq 0$. As a consequence, wave breaking can occur in collisions, which we will now show for the case $N = 2$.

4.2. Wave breaking in collisions between peakons and anti-peakons. For $N = 2$, the system (105)–(106) describing 2-peakon solutions

$$u = \alpha_1 e^{-|x-\beta_1|} + \alpha_2 e^{-|x-\beta_2|} \quad (108)$$

takes a simple form. First, the constant of motion (107) can be used to express the relative separation $|\beta_{1,2}| = |\beta_1 - \beta_2|$ in terms of the two amplitudes α_1 and α_2 through the relation

$$e^{-|\beta_{1,2}|} = \frac{H - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha_2}. \quad (109)$$

Then, the equations of motion for the two positions β_1 and β_2 and the two amplitudes α_1 and α_2 are given by

$$\beta_1' = A_1^p, \quad (110)$$

$$\beta_2' = A_2^p, \quad (111)$$

$$\alpha_1' = \frac{1}{2} \text{sgn}(\beta_{1,2}) A_1^{p-1} (H - \alpha_1^2 - \alpha_2^2), \quad (112)$$

$$\alpha_2' = -\frac{1}{2} \text{sgn}(\beta_{1,2}) A_2^{p-1} (H - \alpha_1^2 - \alpha_2^2), \quad (113)$$

with

$$A_1 = \frac{H + \alpha_1^2 - \alpha_2^2}{2\alpha_1}, \quad A_2 = \frac{H + \alpha_2^2 - \alpha_1^2}{2\alpha_2}, \quad (114)$$

and

$$\beta_{1,2} = \beta_1 - \beta_2. \quad (115)$$

If another constant of motion could be found for this system, then the system could be reduced to two separated ODEs for the two amplitudes, plus two quadratures for the two positions, which would allow the general solution to be obtained. Even without another constant of motion, it is still possible to do a qualitative analysis of all solutions by studying the phase plane (α_1, α_2) of the coupled ODEs (112)–(113) for the amplitudes.

We start from the relation (109), which imposes inequalities on the amplitudes,

$$0 \leq \frac{H - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha_2} \leq 1. \quad (116)$$

For a given value of $H > 0$, these two inequalities define the domain for all 2-peakon solutions in the phase plane (α_1, α_2) . The boundary of the domain corresponds to the two equalities

$$\alpha_1^2 + \alpha_2^2 = H, \quad |\beta_{1,2}| = \infty \quad (117)$$

and

$$(\alpha_1 + \alpha_2)^2 = H, \quad |\beta_{1,2}| = 0 \quad (118)$$

which consist of a circle and two parallel lines. The circle comprises the equilibrium points of the amplitude ODEs (112)–(113) in the phase plane. Each point on the circle is a limit of a 2-peakon solution describing an asymptotic superposition of two 1-peakon solutions, in which the amplitudes are constant and the positions are infinitely separated. The lines each constitute a degenerate 2-peakon solution in which the two positions coincide and the sum of the two amplitudes is constant, describing a peakon solution

$$u(t, x) = \sqrt{H} \exp(-|x - \sqrt{H}^p t|) \quad (119)$$

in the case of the upper line, and an anti-peakon solution

$$u(t, x) = -\sqrt{H} \exp(-|x - (-\sqrt{H})^p t|) \quad (120)$$

in the case of the lower line.

The entire solution domain divides into four parts which are related by a reflection symmetry $(\alpha_1, \alpha_2) \longleftrightarrow (-\alpha_1, -\alpha_2)$. One part of the domain is given by the points lying between the circle (117) and the upper line (118) in the first quadrant, which comprises all solutions describing two peakons. There is a counterpart given by the points lying between the circle (117) and the lower line (118) in the third quadrant, which comprises all solutions describing two anti-peakons. The two other parts of the domain comprise all solutions describing a

peakon and an anti-peakon. These parts are given by the points between the segments of the upper and lower lines that lie outside of the circle.

Within this solution domain in the phase plane, the flow defined by the amplitude ODEs (112)–(113) depends on the nonlinearity power p and the sign of the separation $\beta_{1,2}$. We are interested in flows that describe a collision between a peakon and an anti-peakon. This condition can be used to determine $\text{sgn}(\beta_{1,2})$ at $t = 0$ at each point in the phase plane by considering the ODE

$$\beta_{1,2}' = A_1^p - A_2^p \quad (121)$$

for the separation. If $\beta_{1,2}' > 0$, then the relative separation $|\beta_{1,2}|$ between the peakon and anti-peakon will be decreasing only if $\beta_{1,2} < 0$. Similarly, if $\beta_{1,2}' < 0$, then the relative separation $|\beta_{1,2}|$ between the peakon and anti-peakon will be decreasing only if $\beta_{1,2} > 0$. Hence, a necessary condition for a collision to occur is that $\beta_{1,2}'$ and $\beta_{1,2}$ have opposite signs during the flow. Since $\beta_{1,2} = 0$ can occur only on the upper and lower lines (118), which are boundaries of the domain in which solutions describe a collision between a peakon and an anti-peakon, we can impose

$$\text{sgn}(\beta_{1,2}) = \text{sgn}(A_2^p - A_1^p) \quad (122)$$

at each point in the phase plane. Note $\text{sgn}(\beta_{1,2}) = 0$ holds iff $A_1 = A_2$ when p is odd, and $A_1 = \pm A_2$ when p is even. The points given by $A_1 = A_2$ in the phase plane consist of the lines (118) and $\alpha_1 = \alpha_2$, while the points given by $A_1 = -A_2$ consist of the lines that are perpendicular to each of those three lines. Consequently, hereafter we will consider initial conditions

$$\alpha_2(0) > 0 \quad (\text{peakon}) \quad \text{and} \quad \alpha_1(0) < 0 \quad (\text{anti-peakon}) \quad (123)$$

and

$$\alpha_2(0) + \alpha_1(0) > 0 \quad (124)$$

without loss of generality. (Note that reversing the sign in the initial condition (124) will correspond to reflecting the flow about the line $\alpha_1 + \alpha_2 = 0$ in the phase plane.)

Under the collision condition (122) and initial conditions (123)–(124), the flow then depends only on the nonlinearity power p . The case $p = 1$, which represents the Camassa-Holm equation, is special, since there is another constant of motion $M = \alpha_1 + \alpha_2 = \text{const.}$ which is given by the total mass (101). This implies that the flow simply consists of parallel lines in the phase plane. In all other cases $p \neq 1$, the flow is no longer given by straight lines and has a much richer structure.

The flows for all even powers $p = 2, 4, \dots$ are qualitatively similar to the case $p = 2$, which represents the Novikov equation. A picture of the phase plane for $p = 4$ is shown in Fig. 1. Clearly, in the second quadrant, the upper line $\alpha_1 + \alpha_2 = \sqrt{H}$ is a stable asymptotic attractor for solutions describing a peakon ($\alpha_2 > 0$) and an anti-peakon ($\alpha_1 < 0$), while the lower line $\alpha_1 + \alpha_2 = -\sqrt{H}$ is an unstable asymptotic attractor. In the fourth quadrant, these behaviours are reversed.

The flows for all other odd powers $p = 3, 5, \dots$ are qualitatively similar to the case $p = 3$ which is shown in Fig. 2. In the second quadrant, both the upper and lower lines $\alpha_1 + \alpha_2 = \pm\sqrt{H}$ are stable asymptotic attractors for solutions describing a peakon ($\alpha_2 > 0$) and an anti-peakon ($\alpha_1 < 0$). The line $\alpha_1 + \alpha_2 = 0$ is an unstable asymptotic attractor. In the fourth quadrant, the behaviour is the same.

In all cases $p \neq 1$, the flow will evolve the initial amplitudes toward a stable attractor line. This evolution is shown in Figs. 3 and 4 for the cases $p = 3$ and $p = 4$, respectively, where

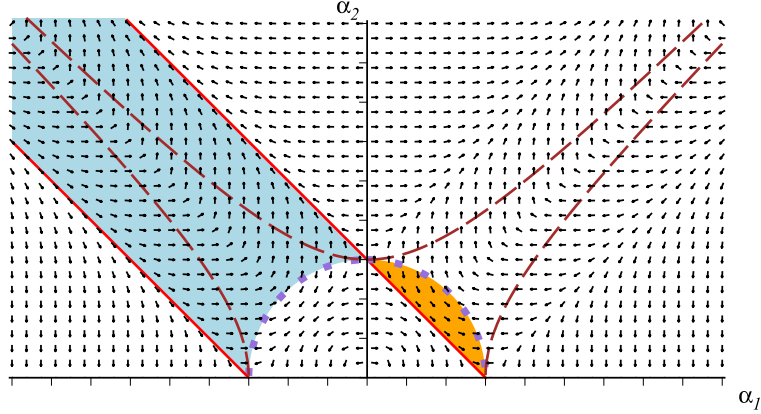


FIGURE 1. Phase plane for collision of peakon and anti-peakon when $p = 4$

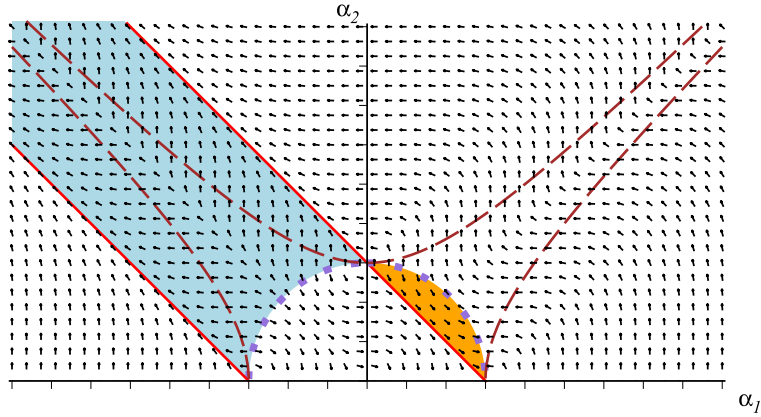


FIGURE 2. Phase plane for collision of peakon and anti-peakon when $p = 3$

the initial positions of the peakon and anti-peakon are chosen to be distinctly separated. We see that the peakon and anti-peakon collide such that their peak amplitudes become closer while the slope at locations x in between the two peaks rapidly increases (without bound) as relative separation between their positions decreases to zero in a finite time. This blow-up in the slope seen in Figs. 5 and 6 is an example of wave breaking.

There is a qualitative explanation of why the blow-up in the slope u_x between the two peaks in a collision solution (108) occurs in a finite time. Consider the asymptotic attractor solution $u(t, x) = \sqrt{H} \exp(-|x - \sqrt{H^p} t|)$ corresponding to the upper line (118). This solution arises from the initial condition $\alpha_1(0) = 0$ and $\alpha_2(0) = \sqrt{H}$. The amplitude ODEs (112)–(113) yield

$$\alpha_1 = \frac{-\sqrt{H}}{\exp(2\sqrt{H^p}(T-t)) - 1}, \quad \alpha_2 = \frac{\sqrt{H} \exp(2\sqrt{H^p}(T-t))}{\exp(2\sqrt{H^p}(T-t)) - 1} \quad (125)$$

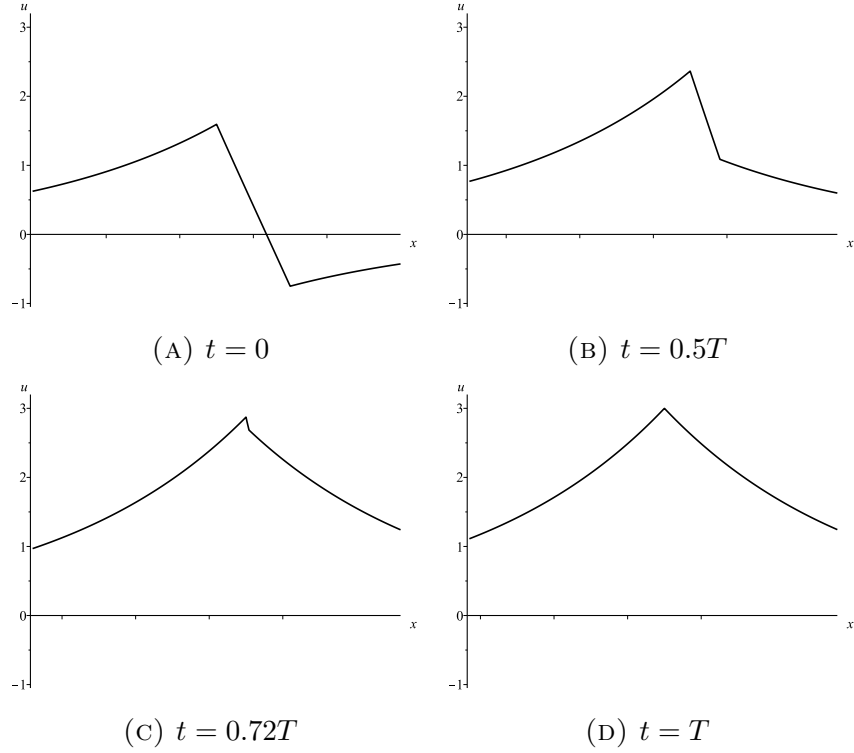


FIGURE 3. Collision of peakon and anti-peakon in the case $p = 3$ ($\alpha_1(0) = -3.5$, $\alpha_2(0) = 4$)

whereby $\alpha_1 \rightarrow -\infty$ and $\alpha_2 \rightarrow \infty$ as $t \rightarrow \infty$ such that $\alpha_1 + \alpha_2 = \sqrt{H}$ is constant for all $t \geq 0$. Any solution having an initial condition close to $\alpha_1(0) = 0$ and $\alpha_2(0) = \sqrt{H}$ will exhibit a similar long-time behaviour for α_1 and α_2 , as a consequence of continuous dependence of solutions on initial data for the ODEs (112)–(113). Since $\alpha_1 + \alpha_2 \rightarrow \sqrt{H}$, the solution (108) remains continuous and bounded at all x for $t \geq 0$, whereas the slope

$$u = \operatorname{sgn}(\beta_1 - x)\alpha_1 e^{-|x-\beta_1|} + \operatorname{sgn}(\beta_2 - x)\alpha_2 e^{-|x-\beta_2|} \quad (126)$$

has jump discontinuities at $x = \beta_1$ and $x = \beta_2$ and becomes unbounded at $x \rightarrow (\beta_1 + \beta_2)/2$ (with $\beta_1 - \beta_2 \rightarrow 0$) as $t \rightarrow T < \infty$.

The same kind of wave-breaking behaviour can be expected to occur in collisions between peakons and anti-peakons when $N > 2$.

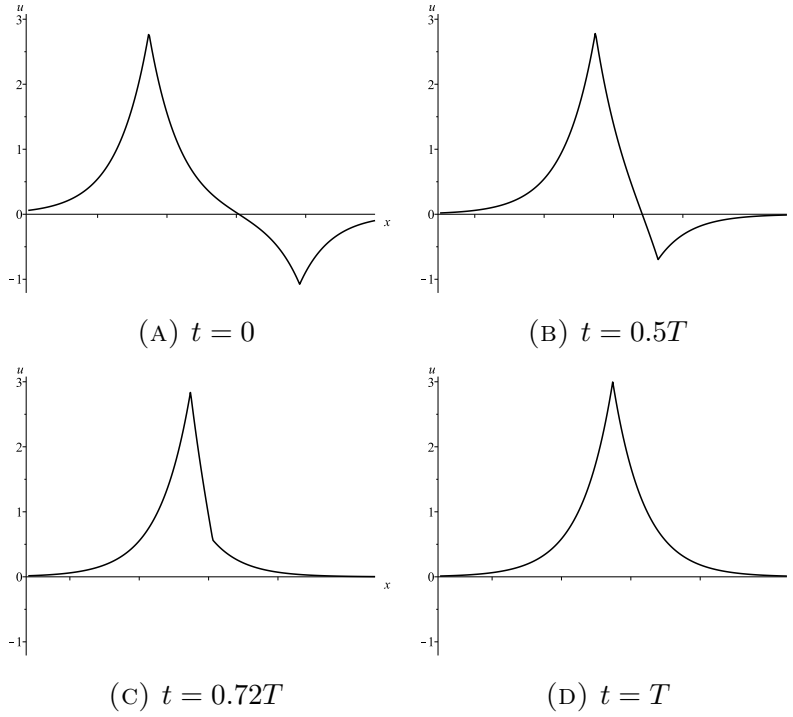


FIGURE 4. Collision of peakon and anti-peakon in the case $p = 4$ ($\alpha_1(0) = -1.1$, $\alpha_2(0) = 2.8$)

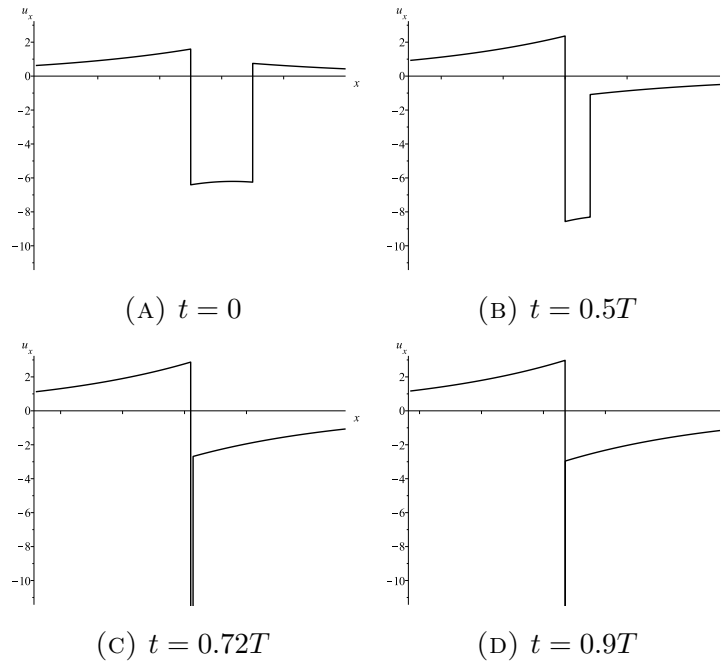


FIGURE 5. Blow-up of slope in collision of peakon and anti-peakon in the case $p = 3$

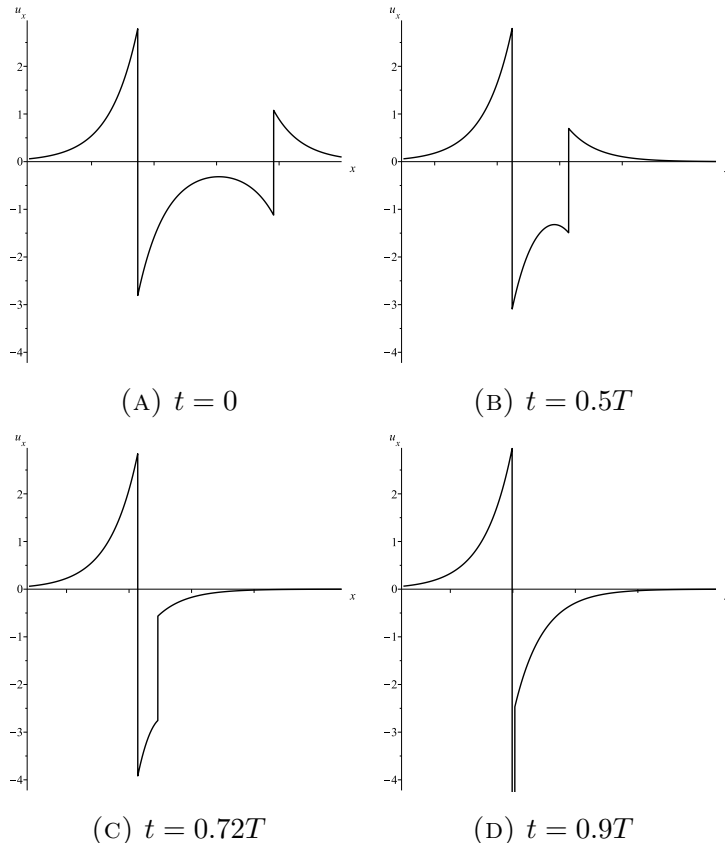


FIGURE 6. Blow-up of slope in collision of peakon and anti-peakon in the case $p = 4$

5. CONCLUDING REMARKS

At first sight, the gCHN equation (10) seems closely analogous to the b -equation (5): both equations unify two integrable equations, possess N -peakon solutions, and exhibit wave breaking phenomena. However, there are important differences. Firstly, the nonlinearities in the b -equation are purely quadratic, whereas the gCHN equation has nonlinearities of degree $p + 1$ and thereby connects two integrable equations with different nonlinearities. Secondly, the H^1 norm of solutions $u(t, x)$ is conserved for the b -equation only if $b = 1$, when the b -equation reduces to the Camassa-Holm equation. In contrast, the H^1 norm is conserved for the gCHN equation for all $p \neq 0$.

In a subsequent work, we will explore further properties of the gCHN equation (10) and its multi-peakon solutions. There are numerous interesting questions. Can a wave-breaking result similar to those for the Camassa-Holm and Novikov equations be established for classical solutions? How will the wave-breaking behaviour depend on p ? In particular, a plausible criteria for wave-breaking is $\liminf_{t \rightarrow T} \inf_{x \in \mathbb{R}} (u^{p-1} u_x) = -\infty$ which generalizes the criteria known [24, 25] in the Camassa-Holm case $p = 1$ and the Novikov case $p = 2$. In another direction, for any p other than these two known integrable cases $p = 1$ and $p = 2$, does the equation have a Hamiltonian formulation or perhaps integrability properties?

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APPENDIX A.

A.1. Lie symmetries. To classify all of the Lie symmetries admitted by the 4-parameter equation (1), we first substitute a general coefficient function $P(t, x, u, u_t, u_x)$ into the symmetry determining equation (20). Next we eliminate u_{txx} , u_{tttx} , u_{txxx} through writing the equation in the solved form

$$u_{txx} = u_t + au^p u_x - bu^{p-1} u_x u_{xx} - cu^p u_{xxx} \quad (127)$$

and doing the same for its differential consequences. The determining equation (20) then splits with respect to u_{xx} , u_{tx} , u_{tt} , u_{xxx} , u_{ttt} , u_{xxxx} into a linear overdetermined system of 10 equations for $P(t, x, u, u_t, u_x)$, a, p, b, c :

$$P_{u_x u_x} = 0, \quad P_{u_t u_x} = 0, \quad P_{u_t u_t} = 0, \quad (128)$$

$$P_{uu_x} = 0, \quad P_{uu_t} = 0, \quad P_{xu_t} = 0, \quad (129)$$

$$2P_{xu} + 2u_x P_{uu} + P_{xxu_x} = 0, \quad (130)$$

$$cu^p (P_{tu_t} - P_{xu_x}) + cpu^{p-1} (u_t P_{u_t} + u_x P_{u_x} - P) - P_{tu_x} = 0, \quad (131)$$

$$(p-1)bu^{p-2} u_x (u_t P_{u_t} + u_x P_{u_x} - P) + 3cu^p (P_{xu} + u_x P_{uu}) - bu^{p-1} (u_x (P_u - P_{tu_t}) + P_x) - P_{tu} - u_t P_{uu} - 2P_{xtu_x} = 0, \quad (132)$$

$$apu^{p-1} u_x (u_t P_{u_t} + u_x P_{u_x} - P) + (au^p u_x + u_t) (u_t P_t + 2u_x P_x) - au^p P_x + cu^p (u_x^3 P_{uuu} + 3u_x^2 P_{xuu} + 3u_x P_{xxu} + P_{xxx}) + bu^{p-1} (u_x^3 P_{uu} + 2u_x^2 P_{xu} + u_x P_{xx}) + u_x^2 (P_{tuu} + u_t P_{uuu}) + 2u_x (P_{xtu} + u_t P_{xuu}) + u_t P_{xxu} + P_{xxt} - P_t = 0. \quad (133)$$

Equation (128) shows that P is a linear function of u_t and u_x , and hence

$$P = \eta - \tau u_t - \xi u_x \quad (134)$$

for some functions $\eta(t, x, u)$, $\tau(t, x, u)$, $\xi(t, x, u)$. After simplifying the remaining equations (129)–(133), we obtain a system of 14 equations

$$\tau_x = 0, \quad \tau_u = 0, \quad \xi_u = 0, \quad \eta_{xuu} = 0, \quad \eta_{tuu} = 0, \quad (135)$$

$$\tau_{tu} - \eta_{uu} = 0, \quad \xi_{xx} - 2\eta_{xu} = 0, \quad b\eta_{uu} + c\eta_{uuu}u = 0, \quad 4\xi_x - \xi_{xxx} = 0, \quad (136)$$

$$\xi_t + c(\xi_x - \tau_t)u^p - cpu^{p-1}\eta = 0, \quad \eta_t - \eta_{txx} + (a\eta_x - c\eta_{xxx})u^p = 0, \quad (137)$$

$$b\eta_{xx} - (a-c)((\xi_x + \tau_t)u + p\eta) = 0, \quad (138)$$

$$4\xi_{tx} - 2\eta_{tu} - 2b\eta_x u^{p-1} + 3c\xi_{xx} u^p = 0, \quad (139)$$

$$3\xi_{tu} - b(p-1)\eta u^{p-2} + b(\xi_x - \tau_t - \eta_u)u^{p-1} - 3\tau_{tu}u^p = 0. \quad (140)$$

We solve this linear overdetermined system by the following steps. First, an integrability analysis of the system of equations (135)–(140) is carried out using the Maple package *rifsimp*, which yields 9 cases. Next, in each case the reduced system of equations is integrated using

the Maple command *pdsolve*. Last, the solutions are merged, which leads to the following classification result.

Proposition A.1. (i) For any $p \neq 0$ and any $(a, b, c) \neq 0$, equation (1) admits no contact symmetries. (ii) The point symmetries admitted by equation (1) for all $p \neq 0$ and all $(a, b, c) \neq 0$ consist of

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = -pt\partial_t + u\partial_u. \quad (141)$$

(iii) Equation (1) admits additional point symmetries only in the following cases:

$$(a) \quad X_{4a} = at\partial_x + \partial_u \quad \text{iff} \quad p = 1, \quad a = c \quad (142)$$

$$(b) \quad X_{4b} = \exp(\pm 2x)(\pm \partial_x + u\partial_u) \quad \text{iff} \quad p = 2, \quad a = 4c, \quad b = 3c \quad (143)$$

The symmetries X_1 , X_2 , X_3 respectively generate one-dimensional point transformation groups consisting of time-translations $t \rightarrow t + \epsilon$, space-translations $x \rightarrow x + \epsilon$, and scalings $t \rightarrow \exp(-b\epsilon)t$, $u \rightarrow \exp(\epsilon)u$, with group parameter $\epsilon \in \mathbb{R}$. The extra symmetry X_{4a} generates the one-dimensional point transformation group $u \rightarrow u + \epsilon$, $x \rightarrow x + \epsilon at$, which is a Galilean boost, and the other extra symmetry X_{4b} generates the one-dimensional point transformation group $u \rightarrow (1 - 2\epsilon \exp(\pm 2x))^{1/2}u$, $x \rightarrow x \pm \frac{1}{2} \ln(1 - 2\epsilon \exp(\pm 2x))$, which is a non-rigid dilation.

A symmetry analysis of particular equations in the 4-parameter family (1) can be found in Refs. [26, 27, 28].

A.2. Low-order multipliers. To classify all 1st-order multipliers admitted by the 4-parameter equation (1), we first substitute the expression (31) into the determining equation (22), which splits into a linear overdetermined system of 5 equations for $Q(t, x, u, u_t, u_x), p, a, b, c$. The system contains the equations

$$Q_{u_t} = 0, \quad Q_{u_x} = 0 \quad (144)$$

which yield

$$Q = Q_0(t, x, u). \quad (145)$$

After the remaining 3 equations are split with respect to u_t and u_x , we obtain the following system of 8 equations

$$Q_{0xu} = 0, \quad Q_{0uu} = 0, \quad (146)$$

$$(p-1)(3pc-2b)Q_{0x} = 0, \quad p((p+1)c-b)Q_{0u} = 0, \quad (3pc-b)Q_{0xx} = 0, \quad (147)$$

$$(p-1)(pc-b)Q_0 + (2pc-b)Q_{0u}u = 0, \quad (148)$$

$$Q_{0tu} + (3pc-b)Q_{0x}u^{p-1} = 0, \quad (149)$$

$$Q_{0xxt} - Q_{0t} + (cQ_{0xxx} - aQ_{0x})u^p = 0. \quad (150)$$

We solve this linear overdetermined system by the same three steps used in solving the symmetry system (135)–(140). This yields the five distinct cases presented in parts (i) and (ii) of Proposition 2.1.

Finally, by splitting and simplifying the determining equation (22) for second-order multipliers of the form (32), we obtain a linear overdetermined system of 13 equations for

$Q(u, u_x, u_{xx})$, a, p, b, c . One of the equations in this system is given by

$$Q_{u_x} = 0 \quad (151)$$

which yields

$$Q = Q_0(u, u_{xx}). \quad (152)$$

The remaining 10 equations then split with respect to u_x , leading to a system of 6 equations

$$Q_{0uu} = 0, \quad Q_{0uu_{xx}} = 0, \quad Q_{0u_{xx}u_{xx}} = 0, \quad (153)$$

$$(p-1)(pc-2b)Q_{0u_{xx}} = 0, \quad (154)$$

$$p(aQ_{0u_{xx}} + ((p+1)c-b)Q_{0u}) = 0 \quad (155)$$

$$(p-1)((pc-b)Q_0 - u_{xx}Q_{0u_{xx}} - (pc-b)Q_{0u}u) = 0. \quad (156)$$

Solving this linear overdetermined system by the same steps used in solving the multiplier system (146)–(150), we obtain the two distinct cases presented in part (iii) of Proposition 2.1.

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