ON THE RADIUS OF SPATIAL ANALYTICITY FOR THE MODIFIED KAWAHARA EQUATION ON THE LINE

GERSON PETRONILHO & PRISCILA LEAL DA SILVA

Abstract. First, by using linear and trilinear estimates in Bourgain type analytic and Gevrey spaces, the local well-posedness of the Cauchy problem for the modified Kawahara equation on the line is established for analytic initial data $u_0(x)$ that can be extended as holomorphic functions in a strip around the x-axis. Next we use this local result and a Gevrey approximate conservation law to prove that global solutions exist. Furthermore, we obtain explicit lower bounds for the radius of spatial analyticity r(t) given by $r(t) \geq ct^{-(4+\delta)}$, where $\delta > 0$ can be taken arbitrarily small and c is a positive constant.

1. Introduction and main result

In this paper we consider the Cauchy problem for the modified Kawahara equation on the line

$$\begin{cases} u_t + \alpha u_{xxxxx} + \beta u_{xxx} + \gamma u_x + \partial_x u^3 = 0, \quad t, x \in \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^2 \mapsto \mathbb{R}$ is a real-valued function and α, β, γ are arbitrary real constants with $\alpha \neq 0$. Equation (1.1) arises in the study of water waves with dimensionless magnitude of surface tension and Weber numbers ϵ next to 1/3, see [1, 7, 10, 13, 14, 15]. It can be considered a generalization of the well-known modified KdV equation

$$u_t + u_{xxx} + u^2 u_x = 0, \tag{1.2}$$

a dispersive equation employed in the study of phenomena such as acoustic waves in inharmonic media and magneto-hydrodynamic waves in collisionless plasma, see [9].

Well-posedness of the Cauchy problem for the modified Kawahara equation on the line (1.1) in Sobolev spaces has been studied by many authors. For instance, Jia and Huo [10] used the Fourier restriction norm to show that (1.1) is locally well-posed in $H^s(\mathbb{R})$ for $s > -1/4$ and Yan, Li and Yang [20] showed the existence of global solutions in $H^s(\mathbb{R})$ for $s > -3/22$.

The first novelty in this paper is the study of the problem of global well-posedness for initial data $u_0(x)$ that are analytic on the line and can be extended as holomorphic functions in a strip around the x-axis. A class of analytic functions suitable for our analysis is the analytic Gevrey class $G^{\sigma,s}(\mathbb{R})$ introduced by Foias and Temam [5], which may be defined as

$$G^{\sigma,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \|f\|^2_{G^{\sigma,s}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\sigma|\xi|}(1+|\xi|)^{2s}|\hat{u}(\xi)|^2 d\xi < \infty \right\}$$

Date: September 12, 2018. *Corresponding author: gersonpetro@gmail.com.

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for $\sigma \geq 0$ and $s \in \mathbb{R}$. In the particular case where $\sigma = 0$, the space $G^{0,s}(\mathbb{R})$ is reduced to the Sobolev space $H^s(\mathbb{R})$, while for $\sigma > 0$ we have the following characterization, see [12]:

Proposition 1.1. (Paley-Wiener Theorem) *Let $\sigma > 0$ and $s \in \mathbb{R}$. Then $f \in G^{\sigma,s}(\mathbb{R})$ if and only if it is the restriction to the real line of a function F which is holomorphic in the strip $\{x+iy : x, y \in \mathbb{R},\ |y| < \sigma\}$ and satisfies*

$$\sup_{|y|<\sigma} \|F(x+iy)\|_{H^s_x(\mathbb{R})} < \infty.$$

In the view of the Paley-Wiener Theorem, it is natural to take initial data in $G^{\sigma,s}(\mathbb{R})$ and obtain a better understanding of the behavior of solution as we try to extend it globally in time. It means that given $u_0 \in G^{\sigma,s}(\mathbb{R})$ for some initial radius $\sigma > 0$ we want to estimate the behavior of the radius of analyticity $\sigma(T)$ as time T goes. This is our second novelty and main goal in this paper.

To achieve this goal first we prove local well-posedness in the space $G^{\sigma,s}(\mathbb{R})$ with $\sigma > 0$ and $s > -\frac{1}{4}$, i.e., the local solution is analytic in the spatial variable. Next we use this local result and a Gevrey approximate conservation law to gradually extend the local solution for all time. Furthermore we obtain explicit lower bounds on the radius of spatial analyticity r(t) at any time $t \geq 0$, which is given by $r(t) \geq ct^{-(4+\delta)}$, where $\delta > 0$ can be taken arbitrarily small and c is a positive constant, that will be described more precisely later.

Our first new result corresponds to local well-posedness, with a bound for the lifespan:

Theorem 1.1. *Let $s > -1/4$ and $\sigma > 0$. Then for any initial data $u_0 \in G^{\sigma,s}(\mathbb{R})$, there exists a positive time $T_{\sigma,s} \doteq T_{\sigma,s,u_0}$, depending only on σ, s and u_0, such that the Cauchy problem (1.1) is locally well-posed in $C([-T_{\sigma,s}, T_{\sigma,s}], G^{\sigma,s}(\mathbb{R}))$. Furthermore, the solution u satisfies the bound*

$$\|u(t)\|_{G^{\sigma,s}(\mathbb{R})} \leq c\|u\|_{X^{\sigma,s,b}(\mathbb{R}^2)} \leq 2C\|u_0\|_{G^{\sigma,s}(\mathbb{R})},\ |t| \leq T_{\sigma,s} \tag{1.3}$$

and

$$T_{\sigma,s} = \frac{1}{(2 + 2^{4\epsilon}16C^5\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})})^{1/\epsilon}}, \tag{1.4}$$

for certain constants $C > 0$ and some $b = \frac{1}{2} + \epsilon$, with $0 < \epsilon < \frac{1}{25}$.

Thus this result shows that for local-in-time the radius of analyticity remains constant. Our next new and main result for modified Kawahara equation yields an estimate on how the width of the strip of the radius of the spatial analyticity decay with time.

Theorem 1.2. *Given $\sigma_0 > 0$, $s > -1/4$ and an initial data $u_0 \in G^{\sigma_0,s}(\mathbb{R})$, the solution u obtained in Theorem 1.1 extends globally in time, and for any $T > 0$, we have*

$$u \in C([-T,T], G^{\sigma(T),s}(\mathbb{R}))$$

with $\sigma(T) = \min\{\sigma_0; cT^{-(4+\delta)}\}$, where $\delta > 0$ can be taken arbitrarily small and c is a positive constant.

The method used here for proving lower bounds on the radius of analyticity was introduced in [18] in the context of the 1D Dirac-Klein-Gordon equations. It was applied to the on line KdV equation in [16] improving an earlier result of Bona et al. [2], to the dispersion-generalized periodic KdV equation in [8] and to the quartic generalized KdV equation on the line in [17].

The remaining of the paper is organized as follows. In Sections 2, 3 and 4 we define the function spaces needed and present all the auxiliary estimates that will be employed in the remaining sections.

In Section 5 we prove Theorem 1.1 using the standard contraction method. Section 6 proves the existence of a fundamental approximate conservation law. In section 7, by combining the approximate conservation law with Theorem 1.1 and applying them repeatedly, we can glue intervals in a way to gradually extend the local solution in time to finally prove Theorem 1.2.

2. Function spaces and auxiliary results

In this section we will present the elementary spaces and lemmas used for the proofs of our theorems. Our first step is to recall the embedding property of Gevrey spaces (see [11] p.460): for all $0 < \sigma' < \sigma$ and $s, s' \in \mathbb{R}$, we have

$$G^{\sigma,s}(\mathbb{R}) \subset G^{\sigma',s'}(\mathbb{R}), \tag{2.1}$$

i.e., $\|\varphi\|_{G^{\sigma',s'}(\mathbb{R})} \leq C_{\sigma,\sigma',s,s'}\|\varphi\|_{G^{\sigma,s}(\mathbb{R})}$.

As in Grujić and Kalisch [6] we consider a space that is a hybrid between the analytic Gevrey space and a space of the Bourgain-type. More precisely, for $\sigma \geq 0, s \in \mathbb{R}$ and $b \in [-1,1]$ define $X^{\sigma,s,b}(\mathbb{R}^2)$ to be the Banach space equipped with the norm

$$\|u\|^2_{X^{\sigma,s,b}(\mathbb{R}^2)} = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\sigma|\xi|}(1+|\xi|)^{2s}(1+|\tau-h(\xi)|)^{2b}|\widehat{u}(\tau,\xi)|^2 d\tau d\xi,$$

where $h(\xi) = -\alpha\xi^5 + \beta\xi^3 - \gamma\xi$ and $\widehat{u}(\tau,\xi) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(t\tau+x\xi)}u(t,x)dtdx$.

For $\sigma = 0$, $X^{\sigma,s,b}(\mathbb{R}^2)$ coincides with the space $X^{s,b}(\mathbb{R}^2)$ introduced by Bourgain [3], and Kenig, Ponce and Vega [14]. The norm of $X^{s,b}(\mathbb{R}^2)$ is explicitly given by

$$\|u\|^2_{X^{s,b}(\mathbb{R}^2)} = \int_{\mathbb{R}}\int_{\mathbb{R}} (1+|\xi|)^{2s}(1+|\tau-h(\xi)|)^{2b}|\widehat{u}(\tau,\xi)|^2 d\tau d\xi.$$

All the well-known properties of the standard $X^{s,b}(\mathbb{R}^2)$ spaces carry over to the Gevrey-modified spaces by making use of the substitution $u \to e^{\sigma|D_x|}u$.

The next lemma shows that $X^{\sigma,s,b}(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}; G^{\sigma,s}(\mathbb{R}))$. For $\sigma = 0$ the proof can be found, for instance, in [19], section 2.6 and for $\sigma > 0$ we suggest to the reader the reference [6].

Lemma 2.1. *Let $s \in \mathbb{R}$ and $b > 1/2$. Then for $u \in X^{\sigma,s,b}(\mathbb{R}^2)$ we have*

$$|u|_{C_{(\mathbb{R}),\sigma,s}} \doteq \sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_{G^{\sigma,s}(\mathbb{R})} \leq L\|u\|_{X^{\sigma,s,b}(\mathbb{R}^2)}, \tag{2.2}$$

where $L > 0$ depends only b and, without loss of generality, we can assume that $L > 1$.

Remark 2.1. This will be of importance due to the fact that we will show that, given an initial data $u_0(x) \in G^{\sigma,s}(\mathbb{R})$ there is a unique solution $u \in X^{\sigma,s,b}(\mathbb{R}^2)$ to the Cauchy problem (1.1), for a certain $b > 1/2$ and therefore there is a solution to the Cauchy problem (1.1), $u \in C([-T,T], G^{\sigma,s}(\mathbb{R}))$.

3. Linear estimates

Now we consider the linear Cauchy problem

$$\begin{cases} u_t + \alpha u_{xxxxx} + \beta u_{xxx} + \gamma u_x = F(t,x) \\ u(0,x) = u_0(x). \end{cases} \tag{3.1}$$

By using Duhamel's formula we may write the solution

$$u(t,x) = W(t)u_0(x) + \int_0^t W(t-t')F(t',x)dt',$$

where $W(t) = e^{-tL}$, L is the linear operator $L = \alpha\partial_x^5 + \beta\partial_x^3 + \gamma\partial_x$ and

$$W(t)\varphi(x) = \int_{\mathbb{R}} e^{i\{x\xi + (-\alpha\xi^5 + \beta\xi^3 - \gamma\xi)t\}} \widehat{\varphi}(\xi) d\xi. \tag{3.2}$$

Let $\psi_1 \in C^\infty(\mathbb{R},\mathbb{R})$, with $0 \le \psi_1 \le 1$, such that

$$\psi_1(t) = \begin{cases} 1, & |t| \le 1, \\ 0, & |t| \ge 2, \end{cases}$$

and for $0 < \delta < 1$ we set $\psi_\delta(t) = \psi_1(t/\delta)$.

Lemma 3.1. *For all $s \in \mathbb{R}, \sigma > 0$ and $b > \frac{1}{2}$ we have*

$$\|\psi_1(t)W(t)u_0\|_{X^{\sigma,s,b}(\mathbb{R}^2)} \le C\|u_0\|_{G^{\sigma,s}(\mathbb{R})},$$

where $C > 0$ depends on ψ_1.

Proof. The proof follows the lines of the proof of Lemma 3.1 in [14]. ∎

Lemma 3.2. *For all $s \in \mathbb{R}, \sigma > 0$, $b \in (1/2, 1)$ and $0 < \delta < 1$, we have*

$$\left\| \psi_\delta(t) \int_0^t W(t-t')F(t')dt' \right\|_{X^{\sigma,s,b}(\mathbb{R}^2)} \le C\delta^{\frac{1}{2}-b} \|F\|_{X^{\sigma,s,b-1}(\mathbb{R}^2)}.$$

Lemma 3.3. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function in time. If $-\frac{1}{2} < b_1 \le b_1' < \frac{1}{2}$, then for any $\delta \in (0,1)$ and $\sigma > 0$ we have*

$$\|\varphi(t/\delta)u\|_{X^{\sigma,s,b_1}(\mathbb{R}^2)} \le C\delta^{b_1'-b_1} \|u\|_{X^{\sigma,s,b_1'}(\mathbb{R}^2)}.$$

The proofs of the Lemma 3.2 and Lemma 3.3 for $\sigma = 0$ can be found in Lemma 2.15 of [10] and in Lemma 2.11 of [19], respectively. These inequalities clearly remain valid for $\sigma > 0$, as one merely has to replace u_0 by $e^{\sigma|D_x|}u_0$, F by $e^{\sigma|D_x|}F$ and u by $e^{\sigma|D_x|}u$ in these results.

4. Trilinear estimate

The next result provides the essential trilinear estimate needed for the proof of Theorem 1.1 and Theorem 1.2.

Lemma 4.1. (Theorem 4.1 of [10]) *Let $s > -1/4$, $b_2 > 1/2$ and $b_2' \in (1/2, 7/10)$. Then*

$$\|\partial_x(v_1v_2v_3)\|_{X^{s,b_2'-1}(\mathbb{R}^2)} \le C\|v_1\|_{X^{s,b_2}(\mathbb{R}^2)}\|v_2\|_{X^{s,b_2}(\mathbb{R}^2)}\|v_3\|_{X^{s,b_2}(\mathbb{R}^2)}$$

for some constant $C > 0$.

Remark 4.1. Setting

$$f_i(\tau,\xi) := (1+|\xi|)^s(1+|\tau-h(\xi)|)^{b_2}\widehat{v_i}(\tau,\xi), \ i = 1,2,3,$$

the estimate of Lemma 4.1 can be rewritten as

$$\begin{aligned}
&\|\partial_x(v_1v_2v_3)\|_{X^{s,b_2'-1}(\mathbb{R}^2)} = \Big\|\frac{\xi(1+|\xi|)^s}{(1+|\tau-h(\xi)|)^{1-b_2'}}\widehat{v_1v_2v_3}(\tau,\xi)\Big\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}\\
&= C\Big\|\frac{\xi(1+|\xi|)^s}{(1+|\tau-h(\xi)|)^{1-b_2'}}\int_{\mathbb{R}^4}\widehat{v_1}(\tau_1,\xi_1)\widehat{v_2}(\tau_2,\xi_2)\widehat{v_3}(\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2)d\xi_1d\xi_2\tau_1d\tau_2\Big\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}\\
&= C\Big\|\frac{\xi(1+|\xi|)^s}{(1+|\tau-h(\xi)|)^{1-b_2'}}\int_{\mathbb{R}^4}\frac{f_1(\tau_1,\xi_1)}{(1+|\xi_1|)^s(1+|\tau_1-h(\xi_1)|)^{b_2}}\\
&\times\frac{f_2(\tau_2,\xi_2)}{(1+|\xi_2|)^s(1+|\tau_2-h(\xi_2)|)^{b_2}} \qquad (4.1)\\
&\times\frac{f_3(\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2)}{(1+|\xi-\xi_1-\xi_2|)^s(1+|\tau-\tau_1-\tau_2-h(\xi-\xi_1-\xi_2)|)^{b_2}}d\xi_1d\xi_2d\tau_1d\tau_2\Big\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}\\
&\leq\ C\|f_1\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}\|f_2\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}\|f_3\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}.
\end{aligned}$$

Corollary 4.1. *Let $\sigma>0$, $s>-1/4$, $b_2>1/2$ and $b_2'\in(1/2,7/10)$. Then*

$$\|\partial_x(u_1u_2u_3)\|_{X^{\sigma,s,b_2'-1}(\mathbb{R}^2)}\leq C\|u_1\|_{X^{\sigma,s,b_2}(\mathbb{R}^2)}\|u_2\|_{X^{\sigma,s,b_2}(\mathbb{R}^2)}\|u_3\|_{X^{\sigma,s,b_2}(\mathbb{R}^2)}$$

for some constant $C>0$.

Proof. This estimate can be restated in the form (4.1) modified by the factor $e^{\sigma|\xi|}$ in the first line of the (4.1) and in the others line we can replace,using the triangle inequality, $e^{\sigma|\xi|}$ by $e^{\sigma|\xi-\xi_1|}e^{\sigma|\xi_1-\xi_2|}e^{\sigma|\xi_2|}$. ∎

Finally, for $\delta>0$ we will need the restrictions of $X^{\sigma,s,b}(\mathbb{R}^2)$ to a time slab $(-\delta,\delta)\times\mathbb{R}$. This space is denoted by $X^{\sigma,s,b}_{(\delta)}(\mathbb{R}^2)$, and is a Banach space when equipped with the norm

$$\|u\|_{X^{\sigma,s,b}_{(\delta)}(\mathbb{R}^2)}=\inf\{\|v\|_{X^{\sigma,s,b}(\mathbb{R}^2)}: v=u \text{ on } (-\delta,\delta)\times\mathbb{R}\}. \qquad (4.2)$$

Lemma 4.2. (Lemmas 5 of [16]) *For $\delta>0$, $\sigma\geq 0, s\in\mathbb{R}, -1/2<b<1/2$ and any time interval $I\subset[-\delta,\delta]$ we have*

$$\|\chi_I u\|_{X^{\sigma,s,b}(\mathbb{R}^2)}\leq C\|u\|_{X^{\sigma,s,b}_{(\delta)}(\mathbb{R}^2)}.$$

5. Proof of Theorem 1.1

5.1. **Existence.** For the proof of local well-posedness in the analytic Gevrey spaces, we will use the standard Banach contraction principle for functions $u\in X^{\sigma,s,b}(\mathbb{R}^2)$ in a given closed ball B. Fix $\sigma>0$, $s>-1/4$ and $u_0\in G^{\sigma,s}(\mathbb{R})$, and define the integral operator

$$\Gamma(u)(t,x):=\psi_1(t)W(t)u_0(x)-\psi_\delta(t)\int_0^t W(t-t')\partial_x u^3(t',x)dt'. \qquad (5.1)$$

If that is useful for the nonlinear estimates, as will be, one can also introduce additional cutoffs in $\partial_x u^3(t',x)$ and consider the equation

$$\Gamma(u)(t,x):=\psi_1(t)W(t)u_0(x)-\psi_\delta(t)\int_0^t W(t-t')\psi_{2\delta}(t')\partial_x u^3(t',x)dt', \qquad (5.2)$$

which is actually identical with (5.1) since $\psi_{2\delta}=1$ on support of ψ_δ.

For $s > -\frac{1}{4}$, $b \in (\frac{1}{2}, 1)$ and $\delta \in (0,1)$ it follows from Lemma 3.1 and Lemma 3.2 that

$$\|\Gamma(u)\|_{X^{\sigma,s,b}(\mathbb{R}^2)} \leq C\|u_0\|_{G^{\sigma,s}(\mathbb{R})} + C\delta^{\frac{1}{2}-b}\|\psi_{2\delta}(t)\partial_x u^3\|_{X^{\sigma,s,b-1}(\mathbb{R}^2)}. \tag{5.3}$$

From now on we shall use the letter C to represent a constant which may change a finite number of times and we fix $b = \frac{1}{2} + \epsilon$, with $0 < \epsilon < \frac{1}{25}$, which means that $b - 1 \in \left(-\frac{1}{2}, -\frac{3}{10}\right)$. Taking $b_1 = b - 1$ and $b_1' = -\frac{1}{2} + 5\epsilon$, we have $-\frac{1}{2} < b_1 = b - 1 = -\frac{1}{2} + \epsilon < b_1' = -\frac{1}{2} + 5\epsilon < -\frac{1}{2} + 5\frac{1}{25} = -\frac{1}{2} + \frac{1}{5} = -\frac{3}{10} < \frac{1}{2}$ and therefore it follows from Lemma 3.3 that

$$\|\psi_{2\delta}(t)\partial_x u^3\|_{X^{\sigma,s,-\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \leq C(2\delta)^{4\epsilon}\|\partial_x u^3\|_{X^{\sigma,s,-\frac{1}{2}+5\epsilon}(\mathbb{R}^2)}. \tag{5.4}$$

Now, choosing $b_2' = \frac{1}{2} + 5\epsilon$ we have $b_2' - 1 = -\frac{1}{2} + 5\epsilon$ and $\frac{1}{2} < b_2' = \frac{1}{2} + 5\epsilon < \frac{1}{2} + 5\frac{1}{25} = \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$ and applying Corollary 4.1 with $u_1 = u_2 = u_3 = u$, $b_2 = b = \frac{1}{2} + \epsilon$ and $b_2' = \frac{1}{2} + 5\epsilon$ we obtain

$$\|\partial_x u^3\|_{X^{\sigma,s,-\frac{1}{2}+5\epsilon}(\mathbb{R}^2)} \leq C\|u\|^3_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)}. \tag{5.5}$$

It follows from (5.3), with $b = \frac{1}{2} + \epsilon$ and $b - 1 = -\frac{1}{2} + \epsilon$, (5.4) and (5.5) that we have

$$\begin{aligned}
\|\Gamma(u)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} &\leq C\|u_0\|_{G^{\sigma,s}(\mathbb{R})} + C\delta^{-\epsilon}\|\psi_{2\delta}(t)\partial_x u^3\|_{X^{\sigma,s,-\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \\
&\leq C\|u_0\|_{G^{\sigma,s}(\mathbb{R})} + C\delta^{-\epsilon}C(2\delta)^{4\epsilon}\|\partial_x u^3\|_{X^{\sigma,s,-\frac{1}{2}+5\epsilon}(\mathbb{R}^2)} \\
&\leq C\|u_0\|_{G^{\sigma,s}(\mathbb{R})} + C\delta^{-\epsilon}C(2\delta)^{4\epsilon}C\|u\|^3_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \\
&= C\|u_0\|_{G^{\sigma,s}(\mathbb{R})} + C^3 2^{4\epsilon}\delta^{3\epsilon}\|u\|^3_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)}.
\end{aligned} \tag{5.6}$$

Let $\mathcal{B}$ be the closed ball given by $\mathcal{B} = \{u \in X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2) : \|u\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \leq 2C\delta^{-\frac{2}{3}\epsilon}\|u_0\|_{G^{\sigma,s}(\mathbb{R})}\}$, where C comes from (5.6). For $u \in B$, from (5.6), we obtain

$$\|\Gamma(u)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \leq C\|u_0\|_{G^{\sigma,s}(\mathbb{R})} + 2^{4\epsilon}8C^6\delta^{\epsilon}\|u_0\|^3_{G^{\sigma,s}(\mathbb{R})}.$$

Thus, if we fix δ such that

$$\delta^{\epsilon} \leq \frac{1}{2 + 2^{4\epsilon}16C^5\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})}}, \tag{5.7}$$

which implies that $2^{4\epsilon}8C^5\delta^{\epsilon}\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})} < \delta^{\epsilon} + 2^{4\epsilon}8C^5\delta^{\epsilon}||u_0||^2_{G^{\sigma,s}(\mathbb{R})} \leq \frac{1}{2}$ and therefore we conclude that

$$\|\Gamma(u)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \leq C\|u_0\|_{G^{\sigma,s}(\mathbb{R})} + \frac{1}{2}C\|u_0\|_{G^{\sigma,s}(\mathbb{R})} < 2C\|u_0\|_{G^{\sigma,s}(\mathbb{R})} \tag{5.8}$$

and since $0 < \delta < 1$ we obtain

$$\|\Gamma(u)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} < 2C\delta^{-\frac{2}{3}\epsilon}\|u_0\|_{G^{\sigma,s}(\mathbb{R})} \tag{5.9}$$

and therefore we have $\Gamma(\mathcal{B}) \subset \mathcal{B}$.

Now observe that

$$\Gamma(u) - \Gamma(v) = -\psi_{\delta}(t)\int_0^t W(t-t')\psi_{2\delta}(t')\partial_x(u^3 - v^3)(t',x)dt'. \tag{5.10}$$

Taking the $X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)$ norm and using Lemma 3.2 and Lemma 3.3 in the same way as in the previous calculations, we obtain

$$\begin{aligned}
\|\Gamma(u)-\Gamma(v)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} &= \left\|\psi_\delta(t)\int_0^t W(t-t')\psi_{2\delta}(t')\partial_x(u^3-v^3)dt'\right\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \\
&\leq C\delta^{-\epsilon}\|\psi_{2\delta}(t')\partial_x(u^3-v^3)\|_{X^{\sigma,s,-\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \\
&\leq C\delta^{-\epsilon}C(2\delta)^{4\epsilon}\|\partial_x(u^3-v^3)\|_{X^{\sigma,s,-\frac{1}{2}+5\epsilon}(\mathbb{R}^2)} \\
&= C^2 2^{4\epsilon}\delta^{3\epsilon}\|\partial_x[(u^2+uv+v^2)(u-v)]\|_{X^{\sigma,s,-\frac{1}{2}+5\epsilon}(\mathbb{R}^2)} \\
&= C^2 2^{4\epsilon}\delta^{3\epsilon}\Big(\|\partial_x[u^2(u-v)]\|_{X^{\sigma,s,-\frac{1}{2}+5\epsilon}(\mathbb{R}^2)} + \|\partial_x[uv(u-v)]\|_{X^{\sigma,s,-\frac{1}{2}+5\epsilon}(\mathbb{R}^2)} \\
&+ \|\partial_x[v^2(u-v)]\|_{X^{\sigma,s,-\frac{1}{2}+5\epsilon}(\mathbb{R}^2)}\Big).
\end{aligned}$$

Now we apply Corollary 4.1, with $b_2=\frac{1}{2}+\epsilon$ and $b_2'=\frac{1}{2}+5\epsilon$, in the last inequalitty and we obtain

$$\begin{aligned}
\|\Gamma(u)-\Gamma(v)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} &\leq C^2 2^{4\epsilon}\delta^{3\epsilon}C\Big(\|u\|^2_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} + \|u\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)}\|v\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \\
&+ \|v\|^2_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)}\Big)\times\|u-v\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)}.
\end{aligned}$$

It follows from the last inequality that for $u,v\in\mathcal{B}$ we have

$$\begin{aligned}
\|\Gamma(u)-\Gamma(v)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} &\leq C^2 2^{4\epsilon}\delta^{3\epsilon}C\big[12C^2\delta^{-\frac{4}{3}\epsilon}\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})}\big]\|u-v\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \\
&= 12C^5 2^{4\epsilon}\delta^{\frac{5}{3}\epsilon}\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})}\|u-v\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)}. \qquad (5.11)
\end{aligned}$$

By using the fact that $0<\delta<1$ and (5.7) we conclude that

$$\begin{aligned}
12C^5 2^{4\epsilon}\delta^{\frac{5}{3}\epsilon}\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})} &\leq 12C^5 2^{4\epsilon}\delta^{\epsilon}\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})} = \frac{3}{4}16C^5 2^{4\epsilon}\delta^{\epsilon}\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})} \\
&< \frac{3}{4}2\delta^{\epsilon} + \frac{3}{4}16C^5 2^{4\epsilon}\delta^{\epsilon}\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})} \\
&= \frac{3}{4}(2\delta^{\epsilon} + 16C^5 2^{4\epsilon}\delta^{\epsilon}\|u_0\|^2_{G^{\sigma,s}(\mathbb{R})}) \leq \frac{3}{4}.
\end{aligned}$$

It follows from (5.11) and from the last inequality that

$$\|\Gamma(u)-\Gamma(v)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \leq \frac{3}{4}\|u-v\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)}, \text{ for } u,v\in\mathcal{B}, \qquad (5.12)$$

hence, Γ is a contraction in $\mathcal{B}$.

Since Γ is a contraction in the Banach space $\mathcal{B}$, it follows that Γ has a unique fixed point $u\in\mathcal{B}$. The function u solves the initial-value problem (1.1) in the space $X^{\sigma,s,b}(\mathbb{R}^2)$, where $b=\frac{1}{2}+\epsilon$ with $0<\epsilon<\frac{1}{25}$, with lifespan $|t|\leq T_{\sigma,s}=\delta$ with δ given in (5.7), by our cut-off function ψ_δ.

It follows from Remark 2.1, with $T_{\sigma,s}=\delta$, that we have proved the existence of a solution to our Cauchy problem which belongs to the space $C([-T_{\sigma,s},T_{\sigma,s}],G^{\sigma,s}(\mathbb{R}))$.

5.2. **Uniqueness.** The uniqueness of the solution in $C([-T_{\sigma,s},T_{\sigma,s}],G^{\sigma,s}(\mathbb{R}))$ will follow if it is known that the solution u belongs to the space $C([0,T_{\sigma,s}],G^{\sigma,s}(\mathbb{R}))$. In fact, by the invariance of (1.1) under the reflection $u(t,x)\to u(-t,-x)$, it suffices to prove this result for positive times, i.e., if u solves (1.1) for $t\in[0,T_{\sigma,s}]$ then $v(t,x)=u(-t,-x)$, $t\in[-T_{\sigma,s},0]$ solves the same problem but now for $t\in[-T_{\sigma,s},0]$.

The uniqueness can be proved by the following standard argument.

Lemma 5.1. *Suppose u and v are solutions to* (1.1) *in $C([0,T_{\sigma,s}],G^{\sigma,s}(\mathbb{R}))$ with $u(0,x)=v(0,x)$ in $G^{\sigma,s}(\mathbb{R})$, where $\sigma>0$ and $s>-\frac{1}{4}$. Then $u=v$.*

Proof. Let $w=u-v$. Then w satisfies the initial-value problem

$$w_t+\alpha w_{xxxxx}+\beta w_{xxx}+\gamma w_x+(u^3-v^3)_x=0,\ \ w(0,x)=0. \tag{5.13}$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})}=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}w^2(t,x)dx=\int_{\mathbb{R}}w(t,x)w_t(t,x)dx=-\int_{\mathbb{R}}w(t,x)\partial_x(u^3-v^3)dx, \tag{5.14}$$

since we have

$$\int_{\mathbb{R}}w(t,x)\partial_x^5w(t,x)dx=\int_{\mathbb{R}}w(t,x)\partial_x^3w(t,x)dx=\int_{\mathbb{R}}w(t,x)\partial_xw(t,x)dx=0.$$

We notice that $u^3-v^3=(u^2+uv+v^2)(u-v)\doteq\psi w$ where $\psi=u^2+uv+v^2$.

Thanks to the equation (5.14) we have

$$\begin{aligned}\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})} &= -2\int_{\mathbb{R}}w(t,x)\partial_x(u^3-v^3)dx\\ &= -2\int_{\mathbb{R}}w(t,x)\partial_x[(u^2+uv+v^2)(u-v)]dx\\ &= -2\int_{\mathbb{R}}w(t,x)\partial_x[\psi(t,x)w(t,x)]dx.\end{aligned}$$

Integrating by parts the last integral we obtain

$$\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})}=-\int_{\mathbb{R}}\partial_x\psi(t,x)w^2(t,x)dx$$

from which we deduce the inequality

$$\left|\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})}\right|\le\|\partial_x\psi\|_{L^\infty([0,T]\times\mathbb{R}}\|w(t)\|^2_{L^2(\mathbb{R})}. \tag{5.15}$$

Since $u,v\in C([0,T_{\sigma,s}];G^{\sigma,s}(\mathbb{R}))$ we have that u and v are continuous in t on the compact set $[0,T_{\sigma,s}]$ and are $G^{\sigma,s}(\mathbb{R})$ in x. Thus, we can conclude that

$$\|\partial_x\psi\|_{L^\infty([0,T_{\sigma,s}]\times\mathbb{R})}\le c<\infty. \tag{5.16}$$

Therefore, from (5.15) and (5.16) we obtain the differential inequality

$$\left|\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})}\right|\le c\|w(t)\|^2_{L^2(\mathbb{R})},\ \ 0\le t\le T_{\sigma,s}.$$

Solving it gives

$$\|w(t)\|^2_{L^2(\mathbb{R})}\le e^c\|w(0)\|^2_{L^2(\mathbb{R})},\ \ 0\le t\le T_{\sigma,s}. \tag{5.17}$$

Since $\|w(0)\|_{L^2(\mathbb{R})}=0$, from (5.17) we obtain that $w(t)=0,\ 0\le t\le T_{\sigma,s}$ or $u=v$. ■

5.3. **Continuous dependence of the initial data.** Let u and v be two solutions of the Cauchy problem with respective initial values u_0 and v_0. By Lemma 2.1 we have

$$\begin{aligned} |u-v|_{C_{(T_{\sigma,s}),\sigma,s}} &= \sup_{t\in[0,T_{\sigma,s}]} \|(u-v)(t,\cdot)\|_{G^{\sigma,s}(\mathbb{R})} \\ &\leq L\|u-v\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} = L\|\Gamma(u)-\Gamma(v)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \end{aligned} \tag{5.18}$$

where $L>1$.

It follows from Lemma 3.1 that

$$\begin{aligned} \|\Gamma(u)-\Gamma(v)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} &\leq C\|u_0-v_0\|_{G^{\sigma,s}(\mathbb{R})} \\ &+ \left\|\psi_\delta(t)\int_0^t W(t-t')\psi_{2\delta}(t')\partial_x(u^3-v^3)(t',x)dt'\right\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)}. \end{aligned}$$

It now follows from (5.10), (5.12) and from the last relation that

$$\|\Gamma(u)-\Gamma(v)\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)} \leq C\|u_0-v_0\|_{G^{\sigma,s}(\mathbb{R})} + \frac{3}{4}\|u-v\|_{X^{\sigma,s,\frac{1}{2}+\epsilon}(\mathbb{R}^2)}. \tag{5.19}$$

Now it follows from (5.18) and (5.19) that

$$|u-v|_{C_{T_{\sigma,s}},\sigma,s} \leq 4LC\|u_0-v_0\|_{G^{\sigma,s}(\mathbb{R})}, \tag{5.20}$$

and therefore the proof of continuous dependence is complete. ■

6. Approximate conservation law

We start by recalling that

$$\|u(t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} u^2(t,x)dx$$

is conserved for a solution u of (1.1), since by using Riemman-Lebesgue's Lemma and integration by parts we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R})} &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2(t,x)dx = \int_{\mathbb{R}} u(t,x)u_t(t,x)dx \\ &= \int_{\mathbb{R}} u(t,x)\big[-\alpha u_{xxxxx}(t,x) - \beta u_{xxx}(t,x) - \gamma u_x(t,x) - (u^3)_x(t,x)\big]dx \\ &= -\int_{\mathbb{R}} u(t,x)(u^3)_x(t,x)dx = \int_{\mathbb{R}} u^3(t,x)u_x(t,x)dx \\ &= \frac{1}{4}\int_{\mathbb{R}} (u^4)_x(t,x)dx = 0. \end{aligned}$$

Our goal in this section is to establish an approximate conservation law for a solution to (1.1) based on the conservation the $L^2(\mathbb{R})$ norm of solutions of the equation. Explicitly, we aim at proving Theorem 6.1.

Theorem 6.1. *Let $\sigma>0$, $0<T_1<T_{\sigma,0}<1$ and $u\in X^{\sigma,0,b}_{(T_1)}(\mathbb{R}^2)$ be the local solution to the Cauchy problem (1.1) restricted to $[0,T_1]\times\mathbb{R}$, where $T_{\sigma,0}$ is given by (1.4), with $s=0$ and $b=\frac{1}{2}+\epsilon$, with $0<\epsilon<\frac{1}{25}$. Given $0<\kappa<1/4$, there exists a constant $C>0$ such that the estimate*

$$\sup_{t\in[0,T_1]}\|u(t)\|^2_{G^{\sigma,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\sigma,0}(\mathbb{R})} + C\sigma^\kappa\|u\|^4_{G^{\sigma,0}(\mathbb{R})} \tag{6.1}$$

holds.

Theorem 6.1 is of fundamental importance as it guarantees, by combining it with Theorem 1.1 and applying them repeatedly, we can glue intervals in a way to gradually extend the local solution in time. This will lead to the global well-posedness of solutions in Gevrey spaces, as in Theorem 1.2.

Next we proceed with technical lemmas that will help us with the proof of Theorem 6.1.

Lemma 6.1. (Lemma 12 of [16]) *For $\sigma > 0$, $\theta \in [0,1]$ and $\alpha, \beta \in \mathbb{R}$, we have the estimate*

$$e^{\sigma|\alpha|}e^{\sigma|\beta|} - e^{\sigma|\alpha+\beta|} \leq [2\sigma \min\{|\alpha|, |\beta|\}]^{\theta} e^{\sigma|\alpha|}e^{\sigma|\beta|}.$$

We will use Lemma 6.1 to prove the following corollary.

Corollary 6.1. *For $\sigma > 0$, $\theta \in [0,1]$ and $\xi, \xi_1, \xi_2 \in \mathbb{R}$, we have*

$$e^{\sigma|\xi_1|}e^{\sigma|\xi_2|}e^{\sigma|\xi-\xi_1-\xi_2|} - e^{\sigma|\xi|} \leq \left[4\sigma \frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{1+|\xi|}\right]^{\theta} e^{\sigma|\xi_1|}e^{\sigma|\xi_2|}e^{\sigma|\xi-\xi_1-\xi_2|}.$$

Proof. If ξ_1, ξ_2 and $\xi - \xi_1 - \xi_2$ have the same sign, there is nothing to prove. Without any loss of generality, suppose $\xi_1 \geq 0$ and $\xi_2 \leq 0$. If $\xi_1 \leq 0$ and $\xi_2 \geq 0$, the change $\tilde{\xi_1} = \xi_2$ and $\tilde{\xi_2} = \xi_1$ will reduce the result to the previous case.

If $\xi - \xi_1 - \xi_2 \geq 0$, writing $\alpha = \xi_1 + (\xi - \xi_1 - \xi_2) = \xi - \xi_2$ we have $\alpha \geq 0$ since $\xi - \xi_1 - \xi_2 \geq 0$ implies that $\xi - \xi_2 \geq \xi_1 \geq 0$. Using Lemma 6.1, we obtain

$$e^{\sigma|\xi_1|}e^{\sigma|\xi_2|}e^{\sigma|\xi-\xi_1-\xi_2|} - e^{\sigma|\xi|} = e^{\sigma|\alpha|}e^{\sigma|\xi_2|} - e^{\sigma|\alpha+\xi_2|} \leq [2\sigma \min\{|\xi-\xi_2|, |\xi_2|\}]^{\theta} e^{\sigma|\xi_1|}e^{\sigma|\xi_2|}e^{\sigma|\xi-\xi_1-\xi_2|}.$$

Analogously, if $\xi - \xi_1 - \xi_2 \leq 0$, then taking $\beta = \xi_2 + (\xi - \xi_1 - \xi_2) = \xi - \xi_1 \leq 0$ we have

$$e^{\sigma|\xi_1|}e^{\sigma|\xi_2|}e^{\sigma|\xi-\xi_1-\xi_2|} - e^{\sigma|\xi|} = e^{\sigma|\beta|}e^{\sigma|\xi_1|} - e^{\sigma|\beta+\xi_1|} \leq [2\sigma \min\{|\xi-\xi_1|, |\xi_1|\}]^{\theta} e^{\sigma|\xi_1|}e^{\sigma|\xi_2|}e^{\sigma|\xi-\xi_1-\xi_2|}.$$

Therefore, for

$$A = \begin{cases} \min\{|\xi-\xi_2|, |\xi_2|\}, & \text{if } \xi - \xi_1 - \xi_2 \geq 0, \\ \min\{|\xi-\xi_1|, |\xi_1|\}, & \text{if } \xi - \xi_1 - \xi_2 \leq 0. \end{cases},$$

we can write

$$e^{\sigma|\xi_1|}e^{\sigma|\xi_2|}e^{\sigma|\xi-\xi_1-\xi_2|} - e^{\sigma|\xi|} \leq [2\sigma A]^{\theta} e^{\sigma|\xi_1|}e^{\sigma|\xi_2|}e^{\sigma|\xi-\xi_1-\xi_2|}.$$

From [16] (page 1014) we know that

$$\min\{|\xi-\xi_1|, |\xi_1|\} \leq 2\frac{(1+|\xi-\xi_1|)(1+|\xi_1|)}{1+|\xi|},$$

and now we can estimate A in the following way. If $\xi - \xi_1 - \xi_2 \geq 0$, then

$$A = \min\{|\xi-\xi_2|, |\xi_2|\} \leq 2\frac{(1+|\xi-\xi_2|)(1+|\xi_2|)}{1+|\xi|}.$$

Now observe that

$$\begin{aligned} 1+|\xi-\xi_2| &= 1+|\xi-\xi_1-\xi_2+\xi_1| \leq 1+|\xi-\xi_1-\xi_2|+|\xi_1| \\ &= (1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|) - |\xi-\xi_1-\xi_2||\xi_1| \\ &\leq (1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|), \end{aligned}$$

which implies that

$$A \leq 2\frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{1+|\xi|}.$$

On the other hand, if $\xi-\xi_1-\xi_2 \leq 0$, we have

$$A = \min\{|\xi-\xi_1|,|\xi_1|\} \leq 2\frac{(1+|\xi-\xi_1|)(1+|\xi_1|)}{1+|\xi|},$$

and the same procedure as above tells us that $1+|\xi-\xi_1| \leq (1+|\xi-\xi_1-\xi_2|)(1+|\xi_2|)$ and we can write

$$A \leq 2\frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{1+|\xi|}.$$

In other words, we conclude that

$$A \leq 2\frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{1+|\xi|}$$

and the result is proven. ∎

Lemma 6.2. *For $\kappa \in [0,1/4)$ there exists $C>0$ such that for all $\sigma>0$ and $u \in X^{\sigma,0,b}(\mathbb{R}^2)$, with $b=\frac{1}{2}+\epsilon$ with $0<\epsilon<\frac{1}{25}$, we have*

$$\|F\|_{X^{0,b-1}(\mathbb{R}^2)} \quad \leq \quad C\sigma^{\kappa}\|u\|^3_{X^{\sigma,0,b}(\mathbb{R}^2)} \tag{6.2}$$

where $F=\partial_x\left[(e^{\sigma|D_x|}u)^3-e^{\sigma|D_x|}(u^3)\right]$.

Proof. Let $G=(e^{\sigma|D_x|}u)^3-e^{\sigma|D_x|}(u^3)$. Then

$$\begin{aligned}\|F\|_{X^{0,b-1}(\mathbb{R}^2)} &= \left\|\frac{\xi}{(1+|\tau-h(\xi)|)^{1-b}}\hat{G}(\tau,\xi)\right\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}\\ &= \Big(\int_{\mathbb{R}^2}\frac{|\xi|^2}{(1+|\tau-h(\xi)|)^{2(1-b)}}|\widehat{G}(\tau,\xi)|^2 d\tau d\xi\Big)^{\frac{1}{2}}.\end{aligned} \tag{6.3}$$

We shall calculate the Fourier transform of G:

$$\begin{aligned}&|\hat{G}(\tau,\xi)| = \Big|\widehat{(e^{\sigma|D_x|}u)^3}-\widehat{e^{\sigma|D_x|}(u^3)}\Big| = C\Big|(e^{\sigma|\xi|}\hat{u}*e^{\sigma|\xi|}\hat{u}*e^{\sigma|\xi|}\hat{u})(\tau,\xi)-e^{\sigma|\xi|}(\hat{u}*\hat{u}*\hat{u})(\tau,\xi)\Big|\\ &= C\Big|\int_{\mathbb{R}^4} e^{\sigma|\xi_1|}\hat{u}(\tau_1,\xi_1)e^{\sigma|\xi_2|}\hat{u}(\tau_2,\xi_2)e^{\sigma|\xi-\xi_1-\xi_2|}\hat{u}(\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2)\\ &-e^{\sigma|\xi|}\hat{u}(\tau_1,\xi_1)\hat{u}(\tau_2,\xi_2)\hat{u}(\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2)d\tau_1 d\tau_2 d\xi_1 d\xi_2\Big|\\ &\leq C\int_{\mathbb{R}^4}\Big(e^{\sigma|\xi_1|}e^{\sigma|\xi_2|}e^{\sigma|\xi-\xi_1-\xi_2|}-e^{\sigma|\xi|}\Big)\Big|\hat{u}(\tau_1,\xi_1)\hat{u}(\tau_2,\xi_2)\hat{u}(\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2)\Big|d\tau_1 d\tau_2 d\xi_1 d\xi_2.\end{aligned}$$

For $\kappa \in [0,1/4) \subset [0,1]$, from Corollary 6.1, we write

$$\begin{aligned}|\hat{G}(\tau,\xi)| \quad &\leq \quad C(4\sigma)^{\kappa}\int_{\mathbb{R}^4}\frac{(1+|\xi-\xi_1-\xi_2|)^{\kappa}(1+|\xi_1|)^{\kappa}(1+|\xi_2|)^{\kappa}}{(1+|\xi|)^{\kappa}}\\ &\times e^{\sigma|\xi_1|}|\hat{u}(\tau_1,\xi_1)|e^{\sigma|\xi_2|}|\hat{u}(\tau_2,\xi_2)|e^{\sigma|\xi-\xi_1-\xi_2|}|\hat{u}(\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2)|d\tau_1 d\tau_2 d\xi_1 d\xi_2.\end{aligned} \tag{6.4}$$

Setting $v = e^{\sigma|D_x|}u$ and $f(\tau,\xi) = (1+|\tau-h(\xi)|)^b\widehat{v}(\tau,\xi)$ we have $e^{\sigma|\xi|}\widehat{u}(\tau,\xi) = \widehat{v}(\tau,\xi) = \frac{f(\tau,\xi)}{(1+|\tau-h(\xi)|)^b}$ and therefore we can write (6.4) as

$$
\begin{aligned}
|\hat{G}(\tau,\xi)| \quad \leq \quad & C(4\sigma)^\kappa \int_{\mathbb{R}^4} \frac{(1+|\xi-\xi_1-\xi_2|)^\kappa(1+|\xi_1|)^\kappa(1+|\xi_2|)^\kappa}{(1+|\xi|)^\kappa} \\
& \frac{|f(\tau_1,\xi_1)|}{(1+|\tau_1-h(\xi_1)|)^b}\frac{|f(\tau_2,\xi_2)|}{(1+|\tau_2-h(\xi_2)|)^b}\frac{|f(\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2)|}{(1+|\tau-\tau_1-\tau_2-h(\xi-\xi_1-\xi_2)|)^b}d\tau_1 d\tau_2 d\xi_1 d\xi_2.
\end{aligned} \tag{6.5}
$$

It follows from (6.3) and (6.5) that

$$
\begin{aligned}
&\|F\|_{X^{0,b-1}(\mathbb{R}^2)} = \left\| \frac{\xi}{(1+|\tau-h(\xi)|)^{1-b}}\hat{G}(\tau,\xi)\right\|_{L^2_{\tau,\xi}(\mathbb{R}^2)} \\
&\leq C(4\sigma)^\kappa\Big[\int_{\mathbb{R}^2}\frac{|\xi|^2}{(1+|\tau-h(\xi)|)^{2(1-b)}}\Big(\int_{\mathbb{R}^4}\frac{(1+|\xi-\xi_1-\xi_2|)^\kappa(1+|\xi_1|)^\kappa(1+|\xi_2|)^\kappa}{(1+|\xi|)^\kappa} \\
&\frac{|f(\tau_1,\xi_1)|}{(1+|\tau_1-h(\xi_1)|)^b}\frac{|f(\tau_2,\xi_2)|}{(1+|\tau_2-h(\xi_2)|)^b}\frac{|f(\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2)|}{(1+|\tau-\tau_1-\tau_2-h(\xi-\xi_1-\xi_2)|)^b}d\tau_1 d\tau_2 d\xi_1 d\xi_2\Big)^2 d\tau d\xi\Big]^{\frac{1}{2}} \\
&= C(4\sigma)^\kappa\Big\|\frac{\xi}{(1+|\tau-h(\xi)|)^{(1-b)}}\int_{\mathbb{R}^4}\frac{(1+|\xi-\xi_1-\xi_2|)^\kappa(1+|\xi_1|)^\kappa(1+|\xi_2|)^\kappa}{(1+|\xi|)^\kappa} \qquad (6.6)\\
&\frac{|f(\tau_1,\xi_1)|}{(1+|\tau_1-h(\xi_1)|)^b}\frac{|f(\tau_2,\xi_2)|}{(1+|\tau_2-h(\xi_2)|)^b}\frac{|f(\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2)|}{(1+|\tau-\tau_1-\tau_2-h(\xi-\xi_1-\xi_2)|)^b}d\tau_1 d\tau_2 d\xi_1 d\xi_2\Big\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}.
\end{aligned}
$$

Now by taking $s = -\kappa \in (-\frac{1}{4}, 0]$ we obtain

$$
\begin{aligned}
\|F\|_{X^{0,b-1}(\mathbb{R}^2)} \leq C(4\sigma)^\kappa\Big\| & \frac{\xi(1+|\xi|)^s}{(1+|\tau-h(\xi)|)^{(1-b)}}\int_{\mathbb{R}^4}\frac{|f(\tau_1,\xi_1)|}{(1+|\xi_1|)^s(1+|\tau_1-h(\xi_1)|)^b} \\
\times & \frac{|f(\tau_2,\xi_2)|}{(1+|\xi_2|)^s(1+|\tau_2-h(\xi_2)|)^b} \\
\times & \frac{|f((\tau-\tau_1-\tau_2,\xi-\xi_1-\xi_2))|}{(1+|\xi-\xi_1-\xi_2|)^s(1+|\tau-\tau_1-\tau_2-h(\xi-\xi_1-\xi_2)|)^b}d\tau_1 d\tau_2 d\xi_1 d\xi_2\Big\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}.
\end{aligned}
$$

Taking $b_2' = b_2 = b = \frac{1}{2}+\epsilon$, with $0 < \epsilon < \frac{1}{25}$ in Remark 4.1 it follows that

$$
\|F\|_{X^{0,b-1}(\mathbb{R}^2)} \leq C\sigma^\kappa\|f\|^3_{L^2_{\tau,\xi}(\mathbb{R}^2)} = C\sigma^\kappa\|v\|^3_{X^{0,b}(\mathbb{R}^2)} = C\sigma^\kappa\|e^{\sigma|D_x|}u\|^3_{X^{0,b}(\mathbb{R}^2)} = C\sigma^\kappa\|u\|^3_{X^{\sigma,0,b}(\mathbb{R}^2)},
$$

concluding the proof. ■

Proof of Theorem 6.1.

Let κ, σ, T_1, b and u as in the statement of Theorem 6.1. We start by defining the auxiliary function $U(x,t) = e^{\sigma|D_x|}u(x,t)$, where $D_x = -i\partial_x$. Since u is real-valued we also have U real-valued. Applying the exponential $e^{\sigma|D_x|}$ to the equation (1.1), it is easily seen that we obtain

$$
U_t + \alpha U_{xxxxx} + \beta U_{xxx} + \gamma U_x + e^{\sigma|D_x|}\partial_x u^3 = 0,
$$

which is equivalent to

$$
U_t + \alpha U_{xxxxx} + \beta U_{xxx} + \gamma U_x + 3U^2U_x = \partial_x U^3 - \partial_x(e^{\sigma|D_x|}u^3).
$$

Therefore, if we set

$$
F = \partial_x\left[\left(e^{\sigma|D_x|}u\right)^3 - e^{\sigma|D_x|}(u^3)\right] \tag{6.7}
$$

we obtain

$$U_t + \alpha U_{xxxxx} + \beta U_{xxx} + \gamma U_x + 3U^2 U_x = F. \tag{6.8}$$

Multiplying the resulting equation (6.8) by U and integrating in $x \in \mathbb{R}$ we obtain

$$\int_{\mathbb{R}} UU_t dx + \int_{\mathbb{R}} \alpha UU_{xxxxx} dx + \beta \int_{\mathbb{R}} UU_{xxx} dx + \gamma \int_{\mathbb{R}} UU_x dx + 3\int_{\mathbb{R}} U^3 U_x dx = \int_{\mathbb{R}} UF dx.$$

By noticing that $\partial_x^j U(x,t) \to 0$ as $|x| \to \infty$ (see [16]) we can use integration by parts obtaining

$$\int_{\mathbb{R}} UU_t dx + \frac{3}{4}\int_{\mathbb{R}} (U^4)_x dx = \int_{\mathbb{R}} UF dx$$

and, therefore we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} U^2 dx = \int_{\mathbb{R}} UU_t dx = \int_{\mathbb{R}} UF dx.$$

Now integrating the last equality with respect to $t \in [0, T_1]$ we obtain

$$\frac{1}{2}\left[\int_{\mathbb{R}} U^2(T_1, x) dx - \int_{\mathbb{R}} U^2(0,x) dx\right] = \int_0^{T_1}\int_{\mathbb{R}} UF dx dt. \tag{6.9}$$

Recalling that

$$\|u(t)\|^2_{G^{\sigma,0}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\sigma|\xi|}|\widehat{u}(t,\xi)|^2 dx = \int_{\mathbb{R}} |\widehat{U}(t,\xi)|^2 dx = \int_{\mathbb{R}} U^2(t,x) dx,$$

where in the last equality we used Plancherel theorem and the fact that we are assuming that the solution u is real valued.

It follows from the last equality and from (6.9) that

$$\begin{aligned}\|u(T_1)\|^2_{G^{\sigma,0}(\mathbb{R})} &= \|u(0)\|^2_{G^{\sigma,0}(\mathbb{R})} + 2\int_{\mathbb{R}^2} \chi_{[0,T_1]}(t) UF dx dt \\ &\leq \|u(0)\|^2_{G^{\sigma,0}} + 2\left|\int_{\mathbb{R}^2} \chi_{[0,T_1]}(t) UF dx dt\right|. \end{aligned} \tag{6.10}$$

The next step is to estimate the integral on the right-hand side of (6.10).

Writing $\chi_{[0,T_1]}(t)UF = (\chi_{[0,T_1]}(t)U)(\chi_{[0,T_1]}(t)F)$ it follows from Parseval's formula that

$$\int_{\mathbb{R}^2} (\chi_{[0,T_1]}(t)U)(\chi_{[0,T_1]}(t)F) dx dt = \int_{\mathbb{R}^2} (\widehat{\chi_{[0,T_1]}(\cdot)U})(\tau,\xi)\overline{(\widehat{\chi_{[0,T_1]}(\cdot)F}(\tau,\xi)} d\xi d\tau.$$

Then, Hölder's inequality yields

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \chi_{[0,T_1]}(t) UF dx dt\right| &= \Big|\int_{\mathbb{R}^2} (1+|\tau - h(\xi)|)^{1-b}(\widehat{\chi_{[0,T_1]}(\cdot)U})(\tau,\xi) \\ &\times \ (1+|\tau - h(\xi)|)^{b-1}\overline{(\widehat{\chi_{[0,T_1]}(\cdot)F})(\tau,\xi)} d\tau d\xi\Big| \\ &\leq \|(1+|\tau - h(\xi)|)^{1-b}(\widehat{\chi_{[0,T_1]}(\cdot)U})(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}^2)} \\ &\times \ \|(1+|\tau - h(\xi)|)^{b-1}(\widehat{\chi_{[0,T_1]}(\cdot)F})(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}^2)} \\ &= \|\chi_{[0,T_1]}(\cdot)U\|_{X^{0,1-b}(\mathbb{R}^2)}\|\chi_{[0,T_1]}(\cdot)F\|_{X^{0,b-1}(\mathbb{R}^2)}. \end{aligned} \tag{6.11}$$

Due to the fact that $b = 1/2 + \epsilon$ with $0 < \epsilon < \frac{1}{25}$ we have both $-1/2 < b-1 < 1/2$ and $-1/2 < 1-b < 1/2$. Therefore, one can use Lemma 4.2 to obtain

$$\|\chi_{[0,T_1]}(\cdot)U\|_{X^{0,1-b}(\mathbb{R}^2)} \leq C\|U\|_{X^{0,1-b}_{(T_1)}(\mathbb{R}^2)} \quad \text{and} \quad \|\chi_{[0,T_1]}(\cdot)F\|_{X^{0,b-1}(\mathbb{R}^2)} \leq C\|F\|_{X^{0,b-1}_{(T_1)}(\mathbb{R}^2)}.$$

Since $0 < T_1 < 1$ and using the fact that $\Psi_{T_1} = 1$ for $t \in [0, T_1]$ and the definition of $\|\cdot\|_{X^{\sigma,s,b}_{(T_1)}(\mathbb{R}^2)}$, (see (4.2)), it follows from (6.11) and from the last relation that

$$\begin{aligned} \left|\int_{\mathbb{R}^2} \chi_{[0,T_1]}(t)UF\,dxdt\right| &\leq C\|U\|_{X^{0,1-b}_{(T_1)}(\mathbb{R}^2)}\|F\|_{X^{0,b-1}_{(T_1)}(\mathbb{R}^2)} \\ &\leq C\|\Psi_{T_1}U\|_{X^{0,1-b}(\mathbb{R}^2)}\|\Psi_{T_1}F\|_{X^{0,b-1}(\mathbb{R}^2)}. \end{aligned} \tag{6.12}$$

Since $-\frac{1}{2} < 1-b < \frac{1}{2}$ and $-\frac{1}{2} < b-1 < \frac{1}{2}$ it follows from Lemma 3.3 that

$$\|\Psi_{T_1}U\|_{X^{0,1-b}(\mathbb{R}^2)} \leq C\|U\|_{X^{0,1-b}(\mathbb{R}^2)} \text{ and } \|\Psi_{T_1}F\|_{X^{0,b-1}(\mathbb{R}^2)} \leq C\|F\|_{X^{0,b-1}(\mathbb{R}^2)}. \tag{6.13}$$

Noticing that

$$\|U\|_{X^{0,1-b}(\mathbb{R}^2)} = ||u||_{X^{\sigma,0,1-b}(\mathbb{R}^2)} \leq \|u\|_{X^{\sigma,0,b}(\mathbb{R}^2)}, \tag{6.14}$$

since we have $1 - b < \frac{1}{2} < \frac{1}{2} + \epsilon = b$, we can conclude that from it and Lemma 6.2 that for any $\kappa \in [0, 1/4)$ there exists a constant C such that

$$\|F\|_{X^{0,b-1}(\mathbb{R}^2)} \leq C\sigma^{\kappa}\|u\|^3_{X^{\sigma,0,b}(\mathbb{R}^2)}. \tag{6.15}$$

Therefore, we conclude from (6.10), (6.12), (6.13), (6.14) and (6.15) that

$$\|u(T_1)\|^2_{G^{\sigma,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\sigma,0}(\mathbb{R})} + C\sigma^{\kappa}\|u\|^4_{X^{\sigma,0,b}(\mathbb{R}^2)}.$$

Finally, by using the condition (1.3) we conclude that

$$\sup_{t\in[0,T_1]} \|u(t)\|^2_{G^{\sigma,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\sigma,0}(\mathbb{R})} + C\sigma^{\kappa}\|u(0)\|^4_{G^{\sigma,0}(\mathbb{R})}. \tag{6.16}$$

The proof is now complete. ■

7. Global extension and radius of analyticity

In this section we will prove Theorem 1.2 using a certain iterate process. Our goal is to show that, given $\sigma_0 > 0$ and $u_0 \in G^{\sigma_0,s}(\mathbb{R})$, for large $T > 0$ it is possible to extend the local solution obtained from Theorem 1.1 to a solution $u(t,\cdot) \in G^{\sigma(T),s}(\mathbb{R})$, with $\sigma(T) = \min\{\sigma_0, cT^{-(4+\delta)}\}$, where $\delta > 0$ can be taken arbitrarily small and c being a constant depending only on σ_0 and u_0, for all $t \in [0,T]$. This will be done in two steps: we will first prove the result for $s = 0$ and then extend it to any $s > -1/4$ and $s \neq 0$ using the embedding given in (2.1).

Given $u_0 \in G^{\sigma_0,s}(\mathbb{R})$, with $s > -\frac{1}{4}$ and $\sigma_0 > 0$, it follows from Theorem 1.1 that there exists an unique solution $u \in C([0,T_{\sigma_0,s}]; G^{\sigma_0,s}(\mathbb{R}))$ to the Cauchy problem (1.1), with

$$T_{\sigma_0,s} = \frac{1}{(2 + 2^{4\epsilon}16C^5\|u_0\|^2_{G^{\sigma_0,s}(\mathbb{R})})^{1/\epsilon}},$$

where $0 < \epsilon < \frac{1}{25}$.

Thus, there is a maximal time

$$T^* \doteq \sup\{T_{\sigma_0,s} : u \in C([0,T_{\sigma_0,s}]; G^{\sigma_0,s}(\mathbb{R})) \text{ solves Cauchy problem (1.1) and satisfies (1.3)}\} \tag{7.1}$$

for which we have $T^* \geq T_{\sigma_0,s} = \frac{1}{(2+2^{4\epsilon}16C^5\|u_0\|^2_{G^{\sigma_0,s}(\mathbb{R})})^{1/\epsilon}}$, $T^* \in (0,\infty]$ and $u \in C([0,T^*); G^{\sigma_0,s}(\mathbb{R}))$.

If $T^* = \infty$, then we are done since the solution is defined for $t \in [0,\infty)$ and from (7.1) we conclude that the radius of spacial analyticity persists for all time, that is

$$r(T) = \sigma_0. \tag{7.2}$$

From now on we will assume that $T^* < \infty$. We need to set the following time-step

$$T_0 = \frac{1}{(2+2^{4\epsilon}32C^5\|u_0\|^2_{G^{\sigma_0,s}(\mathbb{R})})^{1/\epsilon}}. \tag{7.3}$$

Note that $T_{\sigma_0,s} = \frac{1}{(2+2^{4\epsilon}16C^5\|u_0\|^2_{G^{\sigma_0,s}(\mathbb{R})})^{1/\epsilon}} > \frac{1}{(2+2^{4\epsilon}32C^5\|u_0\|^2_{G^{\sigma_0,s}(\mathbb{R})})^{1/\epsilon}} = T_0$.

Since $u \in C([0,T_{\sigma_0,s}]; G^{\sigma_0,s}(\mathbb{R}))$ and $T_0 < T_{\sigma_0,s}$ we have $u \in C([0,T_0]; G^{\sigma_0,s}(\mathbb{R}))$, which in turn implies $u \in C([0,T_0]; G^{\sigma,s}(\mathbb{R}))$ for $\sigma < \sigma_0$, since we have $G^{\sigma_0,s}(\mathbb{R}) \hookrightarrow G^{\sigma,s}(\mathbb{R})$.

In order to prove that the solution u can be extend for any $T \geq T^*$ it is suffices to show that

$$u \in C([0,T]; G^{\sigma(T),s}(\mathbb{R})), \text{ for all}\, T \geq T^*, \tag{7.4}$$

where $\sigma(T) = cT^{-\frac{1}{\kappa}}$ with $c > 0$ depending on u_0 and σ_0.

Case $s = 0$**.** By using Theorem 6.1 and the fact that $\|u(0)\|_{G^{\sigma,0}(\mathbb{R})} \leq \|u(0)\|_{G^{\sigma_0,0}(\mathbb{R})}$, since $\sigma < \sigma_0$ we have

$$\sup_{t\in[0,T_0]} \|u(t)\|^2_{G^{\sigma,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\sigma_0,0}(\mathbb{R})} + C\sigma^\kappa \|u(0)\|^4_{G^{\sigma_0,0}(\mathbb{R})}. \tag{7.5}$$

If we assume that

$$C\sigma^\kappa \|u(0)\|^2_{G^{\sigma_0,0}(\mathbb{R})} \leq 1, \tag{7.6}$$

then it follows from (7.5) that

$$\sup_{t\in[0,T_0]} \|u(t)\|^2_{G^{\sigma,0}(\mathbb{R})} \leq 2\|u(0)\|^2_{G^{\sigma_0,0}(\mathbb{R})} < \infty. \tag{7.7}$$

It follows, by using $\|u(T_0)\|_{G^{\sigma,0}(\mathbb{R})}$ as the initial value and repeating the argument before, that the Cauchy problem (1.1) has an unique solution in $[0,2T_0] \times \mathbb{R}$. In this way we succeed in extending the solution of (1.1) to the time interval $[0,2T_0]$.

The above argument can be repeated for ℓ steps, where ℓ is the maximal positive integer such that $\ell C\sigma^\kappa \|u(0)\|^2_{G^{\sigma_0,0}(\mathbb{R})} \leq 1$. Therefore, the desired assertion follows if we can choose a number ℓ such that $\ell T_0 \geq T$ and $\ell C\sigma^\kappa \|u(0)\|^2_{G^{\sigma_0,0}(\mathbb{R})} \leq 1$. Thus, in order to make these conditions be satisfied we start by recalling that $\frac{T}{T_0} > 1$ and defining $\ell = \left[\frac{T}{T_0}\right] + 1$, where $[x]$ the largest integer less than or equal to x, we have

$$\ell = \left[\frac{T}{T_0}\right] + 1 > \frac{T}{T_0}$$

and therefore $\ell T_0 > T$.

Now with this choice of ℓ we can choose

$$\sigma^\kappa \leq \frac{1}{\left(\left[\frac{T}{T_0}\right] + 1\right) C\|u_0\|^2_{G^{\sigma_0,0}}} < \frac{T_0}{TC\|u_0\|^2_{G^{\sigma_0,0}}},$$

which implies that

$$\sigma \leq \Big(\frac{T_0}{C\|u_0\|^2_{G^{\sigma_0,0}}}\Big)^{\frac{1}{\kappa}} T^{-\frac{1}{\kappa}} \doteq c T^{-\frac{1}{\kappa}}.$$

The general case. For $s > -\frac{1}{4}$ and $s \neq 0$ we will use the embedding given by (2.1), i.e.,

$$G^{\sigma,s}(\mathbb{R}) \subset G^{\sigma',s'}(\mathbb{R}) \text{ for all } 0 < \sigma' < \sigma \text{ and } s, s' \in \mathbb{R},$$

i.e., $\|\varphi\|_{G^{\sigma',s'}(\mathbb{R})} \leq C_{\sigma,\sigma',s,s'} \|\varphi\|_{G^{\sigma,s}(\mathbb{R})}$.

It follows from this embedding that

$$u_0 \in G^{\sigma_0,s}(\mathbb{R}) \subset G^{\frac{\sigma_0}{2},0}(\mathbb{R}).$$

It follows from the case $s = 0$, already proved, that there is a $\tilde{T}_0 > 0$ such that

$$u \in C([0,\tilde{T}_0], G^{\sigma_0/2,0}(\mathbb{R}))$$

and

$$u \in C([0,T], G^{2c_1 T^{-\frac{1}{\kappa}},0}(\mathbb{R})), \text{ for } T \geq \tilde{T}_0,$$

where $c_1 > 0$ depends on u_0, σ_0.

Applying again the above embedding we now conclude that

$$u \in C([0,\tilde{T}_0], G^{\sigma_0/4,s}(\mathbb{R}))$$

and

$$u \in C([0,T], G^{c_1 T^{-\frac{1}{\kappa}},s}(\mathbb{R})) \text{ for } T \geq \tilde{T}_0.$$

The proof of Theorem 1.2 is now complete. ■

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Gerson Petronilho (*Corresponding author*)
Departamento de Matemática
Universidade Federal de São Carlos
São Carlos, SP 13565-905, Brazil
E-mail: *gersonpetro@gmail.com*

Priscila L. da Silva
Departamento de Matemática
Universidade Federal de São Carlos
São Carlos, SP 13565-905, Brazil
E-mail: *pri.leal.silva@gmail.com*